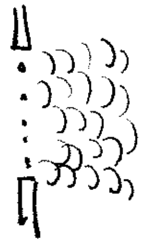
Kirchhoff's Theory of Diffraction

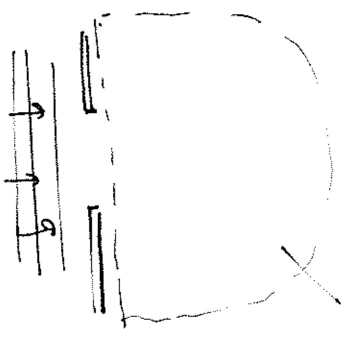
May 11

In the previous lecture, I motivated the idea that the pattern of radiation set up by a hole in a screen should be given by an integral over sources in the hole:



This idea follows from Huygens' principle. But can we derive it rigorously.

We would like to solve the scalar wave equation in a region V bounded by a screen with a hole in it, for the boundary condition that an incoming plane wave is impinging on the hole:



The first step is to write the solution at a point in the interior in terms of the solution at points on the boundary. This can be done as follows: Consider an incoming wave of a definite frequency ω. Then the entire solution will oscillate with frequency ω and we can write it as

$$\psi(t,\vec{x}) = \mathrm{Re}\left[e^{-i\omega t} \psi_0(\vec{x}) \right]$$

$\psi_0(\vec{x})$ satisfies the equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \psi = 0 \Rightarrow \left(-\frac{\omega^2}{c^2} - \nabla^2 \right) \psi_0(\vec{x}) = 0$$

or

$$(-k^2 - \nabla^2)\, \psi_0(\vec{x}) = 0$$

This is called the <u>Helmholtz</u> equation. We met a very similar equation earlier in the course – the static Klein-Gordon equation:

$$(-\nabla^2 + \mu^2)\, \phi = 0$$

We actually derived the <u>Green's</u> <u>function</u> for this equation, the solution of

$$(-\nabla_x^2 + \mu^2)\, G(x;y) = \delta^{(3)}(\vec{x-y})$$

we found

$$G(x;y) = \frac{e^{-\mu|\vec{x-y}|}}{4\pi\,|\vec{x-y}|}$$

Making obvious changes in the formulae, we see that

$$G(x;y) = \frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}$$

is the Green's function that satisfies

$$(-\nabla^2 - k^2)\ G(x;y) = \delta^{(3)}(\vec{x}-\vec{y})$$

Actually, there are several choices for the Green's function of the Helmholtz equation; I have chosen the one that has the property we are looking for that the solution as $|\vec{x}| \to \infty$ is an outgoing spherical wave

$$\frac{e^{-i\omega t}\, e^{ikx}}{x}$$

with no incoming component.

Now let $\vec{x}$ be a point in the interior of V.

For any function $\psi_0(\vec{x})$

$$\int_V d^3y\ \psi_0(\vec{y})\,(-\nabla_y^2 - k^2)\, G(\vec{y};\vec{x})$$

$$= \int d^3y\ \psi_0(\vec{y})\ \delta^{(3)}(\vec{y}-\vec{x})$$

$$= \psi_0(\vec{x})$$

Now impose the condition that $\psi_0(\vec{x})$ is a solution of the Helmholtz equation. Integrating by parts:

$$\psi_0(\vec{x}) = \int_V d^3y \;\; \psi_0(\vec{y}) \left[-\nabla^2 - k^2\right] G(y;x)$$

$$= \int_{\partial V} d^2y \; \hat{n} \cdot \left\{ -\psi_0(\vec{y}) \vec{\nabla} G(y;x) + \vec{\nabla}\psi_0(\vec{y}) \cdot G(\vec{y};\vec{x}) \right\}$$

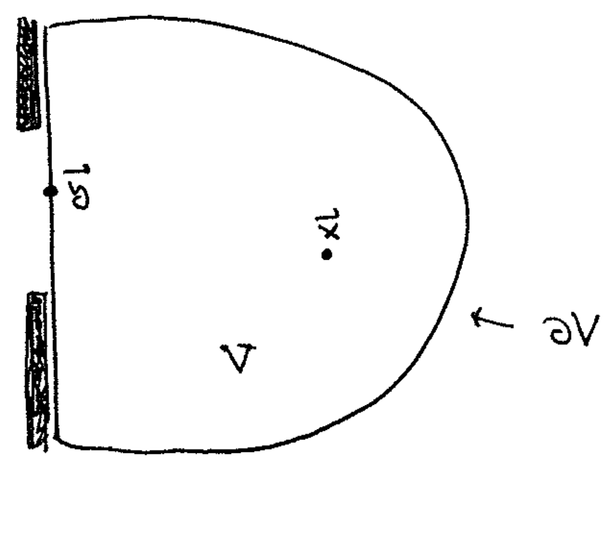
$$+ \int_V d^3y \; \left(-\nabla^2 \psi_0 - k^2 \psi_0\right) G(y;x)$$

The volume term is zero and we have successfully represented $\psi_0(\vec{x})$ by a surface integral.

Now we have:

$$\psi_0(\vec{x}) = \int_{\partial V} d^2y \; \hat{n} \left\{ \frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} \vec{\nabla}_y \psi_0(\vec{y}) - \vec{\nabla}_y\left(\frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}\right) \psi_0(\vec{y}) \right\}$$

The integral is taken over the surface:



We can make further progress by introducing an approximate form for the values of $\psi_0(\vec{y})$ on the boundary:

- In the hole $\psi_0(\vec{y}) \approx$ incoming plane wave

$$\psi_0(\vec{y}) \cong \psi_0 \, e^{ik\hat{z}\cdot\vec{y}}$$

(This approximation will break down near the edges of the hole, but it will be appropriate over most of the hole if the size of the hole $R \gg \lambda$.)

- Behind the screen, away from the hole,

$$\psi_0(\vec{y}) \approx 0$$

- On the surface at ∞, $\psi_0(\vec{y}) = 0$

(or, at least, we will ignore any incoming plane wave contributions coming back from ∞.)

With this set of assumptions:

$$\psi_0(\vec{x}) = \int_{\vec{y} \text{ in the hole}} d^2y \, (-\hat{z})\cdot\left\{ \frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} \vec{\nabla}_y \left(e^{ik\hat{z}\cdot\vec{y}}\right) - \vec{\nabla}_y\left(\frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}\right) e^{ik\hat{z}\cdot\vec{y}} \right\}$$

now $\quad \vec{\nabla}_y \, e^{ik\hat{z}\cdot\vec{y}} = ik\hat{z} \, e^{ik\hat{z}\cdot\vec{y}}$

$$-\vec{\nabla}_y \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} = +\vec{\nabla}_x \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}$$

$$= ik\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}\frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} - \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|^3}(\vec{x}-\vec{y})$$

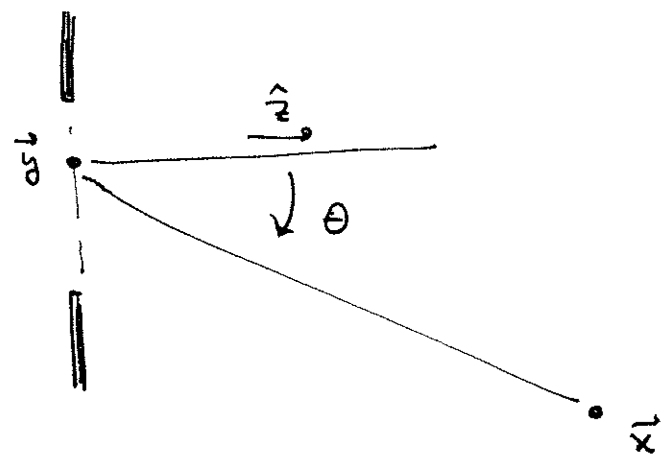
the second term here is smaller than the first by

$$\frac{1}{k|\vec{x}-\vec{y}|} \sim \frac{\lambda}{|\vec{x}-\vec{y}|}$$

so at a point further than a few wavelengths behind the sceen, this term can be neglected. We then have: $\quad (\hat{z}\cdot\vec{y}=0$ at the hole$)$

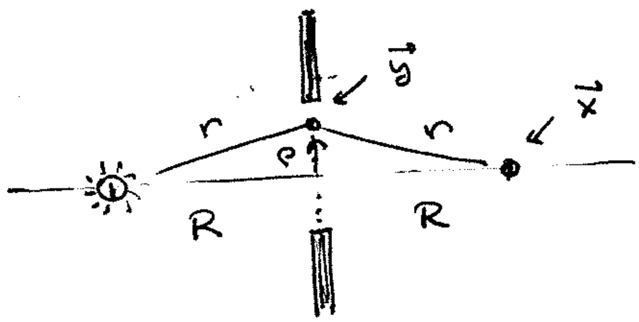
$$\psi_0(\vec{x}) = \frac{-ik\psi_0}{4\pi}\int_{\text{hole}} d^2y \, \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \, (1+\cos\Theta)$$

where $\quad \cos\Theta = \hat{z}\cdot(\widehat{\vec{x}-\vec{y}})$



Aside from the complication of the "obliquity factor" $(1+\cos\theta)$ (due to Stokes), this is exactly the simplest possible Huygens' principle ~~integral~~ over sources filling the hole in the screen.

In the next few lectures, I would like to evaluate this integral formula for the diffracted wave in a number of approximation schemes. First, though, I would like to present one interesting example in which it can be evaluated more or less exactly. This is the situation in which the aperture is a circular disk and we evaluate the field at a point on the axis of symmetry. To simplify even further, I'll take the source of radiation to be a finite distance away, at a position symmetric with the observation point



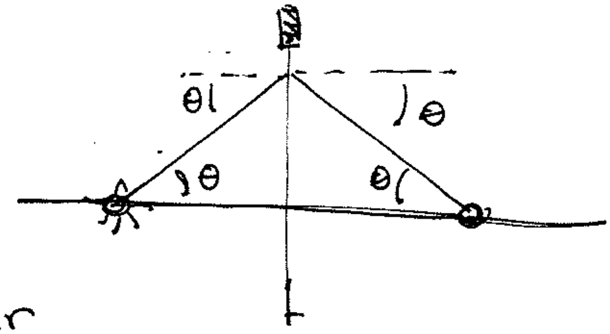
For this setup, the incoming plane wave on p.5 is replaced by a spherical wave

$$\psi_0(\vec{y}) = \psi_0' \frac{e^{ikr}}{r}$$

so that in the first term of the Kirchhoff integral we get

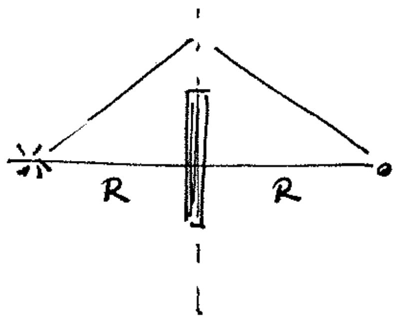
$$\hat{z} \cdot \hat{r} = \cos\Theta$$

rather than $\hat{z} \cdot \hat{z} = 1$. Then



$$\psi(\vec{x}) = -ik \frac{\psi_0'}{4\pi} \int_{\text{hole}} d^2y \, \frac{e^{2ikr}}{r^2} (2\cos\Theta)$$

It is ~~interesting~~ to evaluate this both for a circular hole in a screen and for a circular obstacle:



In each case, I will take the radius to be a. Let's actually consider the obstacle first. Then

$$\psi(\vec{x}) = -ik \frac{\psi_0'}{4\pi} \int_a^\infty d\rho \, \rho \, 2\pi \, \frac{e^{2ikr}}{r^2} \, 2\cos\Theta$$

now $r = [R^2 + \rho^2]^{1/2}$ $\quad \cos\Theta = \frac{R}{r}$ $\quad$ so

$$\psi(\vec{x}) = -ik \frac{\psi_0'}{4\pi} \cdot 4\pi \int_a^\infty d\rho \, \rho \, \frac{R}{r^3} \, e^{2ikr}$$

now $r^2 = \rho^2 + R^2 \Rightarrow dr\, r = d\rho\, \rho$

$$\psi(\vec{x}) = -ik \frac{\psi_0'}{\cdot 1} \int_{\sqrt{a^2+R^2}}^{\infty} dr \frac{R}{r^2} e^{2ikr}$$

$$= -ikR\, \psi_0' \int_{\sqrt{a^2+R^2}}^{\infty} dr \frac{1}{r^2} e^{2ikr}$$

This integral can be approximated nicely by integrating by parts:

$$\int_b^\infty dr \frac{1}{r^2} e^{2ikr} = \int_b dr \frac{1}{r^2} \frac{1}{2ik} \frac{d}{dr}\left(e^{2ikr}\right)$$

$$= \frac{1}{r^2} \frac{1}{2ik} e^{2ikr}\Big|_b^\infty - \int_b^\infty dr \left(-\frac{2}{r^3}\right) \frac{1}{2ik} e^{2ikr}$$

$$= -\frac{1}{b^2} \frac{1}{2ik} e^{2ikb} + \int_b^\infty dr \frac{2}{r^3} \frac{1}{(2ik)^2} \frac{d}{dr}\left(e^{2ikr}\right)$$

$$= -\frac{1}{b^2} \frac{e^{2ikb}}{(2ik)} + \frac{2}{r^3} \frac{1}{(2ik)^2} e^{2ikr}\Big|_b^\infty - \int_b^\infty dr - \frac{2\cdot 3}{r^4} \frac{e^{2ikr}}{(2ik)^2}$$

$$= -\frac{1}{b^2} \frac{e^{2ikr}}{2ik} - \frac{2}{b^3} \frac{e^{2ikr}}{(2ik)^2} + \int dr \frac{3!}{r^4} \frac{1}{(2ik)^3} \frac{d}{dr}\left(e^{2ikr}\right)$$

etc.

Every successive term is smaller by $\frac{1}{b \cdot k} \sim \frac{\lambda}{b}$. So for

$R \sim a \gg \lambda$, we can approximate the integral by the first term and obtain

$$\psi(\vec{x}) = \psi_0' \frac{R}{2(R^2+a^2)} e^{2ik\sqrt{R^2+a^2}}$$

This might be compared to the original spherical wave evaluated at the observation point

$$\psi_0' \frac{1}{2R} e^{2ikR}$$

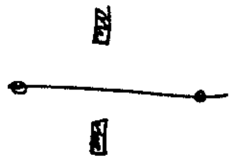
This is an odd conclusion:

$$\frac{I}{I_0} = \left| \frac{\psi(\vec{x})\big|_{\text{behind circular disk}}}{\psi(\vec{x})\big|_{\text{no disk}}} \right|^2 = \left(\frac{R^2}{R^2+a^2} \right)^2$$

In particular, there is always some brightness on the axis behind the disk!

In Born and Wolf, one finds the following interesting footnote: "That a bright spot should appear at the centre of the shadow of a small disc was deduced from Fresnel's theory by S. D. Poisson in 1818. Poisson, who was a member of the committee that reviewed Fresnel's prize memoir, appears to have considered this conclusion contrary to experiment and so refuting Fresnel's theory. However, Arago, another member of the committee, performed the experiment and found that the surprising prediction was correct."

Now return to the case of a circular aperture.



In this case

$$\psi(\vec{x}) = -ikR\,\psi_0' \int_0^a d\rho\,\rho\,\frac{1}{r^3}\,e^{2ikr}$$

$$= -ikR\,\psi_0' \int_R^{[a^2+R^2]^{1/2}} dr\,\frac{1}{r^2}\,e^{2ikr}$$

If $R \gg \lambda$, we can again evaluate the integral by integration by parts:

$$\int_R^{[a^2+R^2]^{1/2}} dr\,\frac{1}{r^2}\,e^{2ikr} = \frac{1}{r^2}\,\frac{1}{2ik}\,e^{2ikr}\Big|_R^{(a^2+R^2)^{1/2}} \cdot \left[1 + O\left(\frac{\lambda}{R}\right)\right]$$

$$\cong \frac{1}{2ik}\left\{ \frac{e^{2ik(a^2+R^2)^{1/2}}}{a^2+R^2} - \frac{e^{2ikR}}{R^2} \right\}$$

Now we have an expression that oscillates as a function of R.

To get a feel for this expression, expand for $R \gg a$

$$\frac{1}{a^2+R^2} = \frac{1}{R^2} - \frac{a^2}{R^4} + \cdots$$

$$k[a^2+R^2]^{\frac{1}{2}} = kR + \frac{ka^2}{2R} + \cdots$$

The corrections from the denominator are down by a^2/R^2, those from the exponent by

$$\frac{ka^2}{R} \sim \frac{a}{\lambda}\frac{a}{R}$$

which is not so small. Keeping the second effect only

$$\psi(\vec{x}) = -\frac{R}{2}\psi_0' \left\{ \frac{e^{2ik[a^2+R^2]^{\frac{1}{2}}}}{a^2+R^2} - \frac{e^{2ikR}}{R^2} \right\}$$

$$\cong -\frac{1}{2R}\psi_0' \, e^{2ikR} \left(e^{ika^2/R} - 1 \right)$$

Again we can compare this to the original spherical wave evaluated at the observation point, $\psi' \frac{1}{2R} e^{2ikR}$. Then

$$\left| \frac{\psi(\vec{x})\big|_{\text{behind circular aperture}}}{\psi\big|_{\text{no screen}}} \right|^2 \cdot \frac{I}{I_0} = \left| e^{ika^2/R} - 1 \right|^2$$

$$= 4 \sin^2 \frac{ka^2}{2R}$$

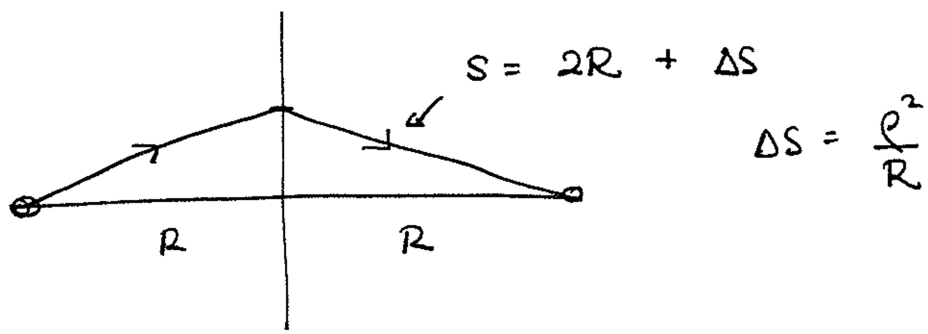
so the intensity oscillates between zero and $(2)^2$ times the original value.

Fresnel found an instructive way to explain what is going on in this example. Go back to p. 11 and make the approximation of the previous page in the integral

$$\psi(\vec{x}) = -ikR\,\psi_0' \int_0^a d\rho\,\rho\,\frac{1}{[R^2+\rho^2]^{3/2}}\,e^{2ik[\rho^2+R^2]^{1/2}}$$

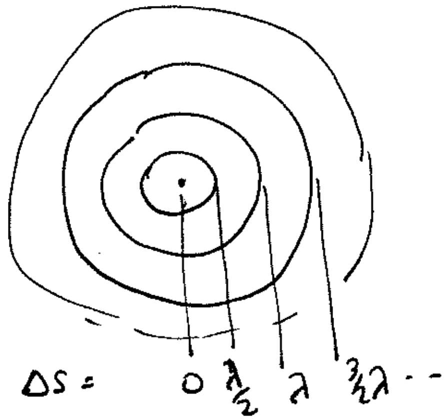
$$= -ikR\,\psi_0'\,\frac{1}{R^3}\,e^{2ikR}\int d\rho\,\rho\;e^{2ik\,\rho^2/2R}$$

The exponent under the ~~integral~~ is the excess path length taken by a ray that goes to radius ρ:



$$\psi(\vec{x}) = -ik\,\frac{1}{R^2}\,\psi_0'\,e^{2ikR}\,\frac{1}{2}\int d(\Delta S)\;e^{ik\,\Delta S}$$

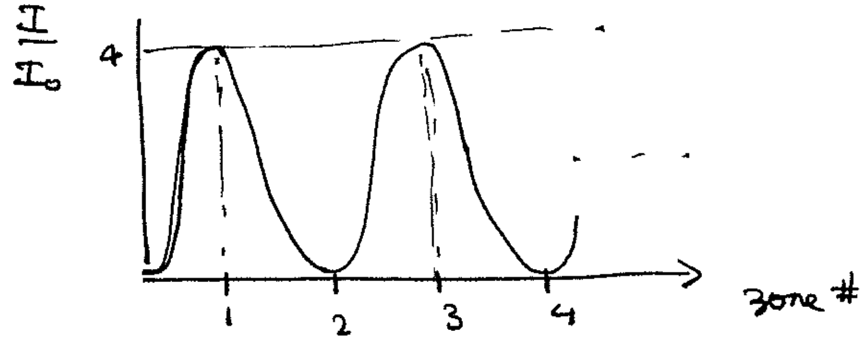
Now, mark circles on the disk at those radii for which $\Delta S = 0,\ \tfrac{1}{2}\lambda,\ \lambda,\ \tfrac{3}{2}\lambda,\ \dots$



The ring corresponding to $\frac{(n-1)\lambda}{2} < \Delta s < \frac{n\lambda}{2}$ is called the <u>nth</u> <u>Fresnel zone</u>. Each Fresnel zone makes a contribution to the amplitude:

$$\int_{(n-1)\lambda/2}^{n\lambda/2} d(\Delta s)\; e^{ik(\Delta s)} = (-1)^{n-1} \int_0^{\lambda/2} d\sigma \; e^{ik\sigma}$$

So the 1st zone gives a coherent contribution, the 2nd zone gives a <u>cancelling</u> contribution, etc.



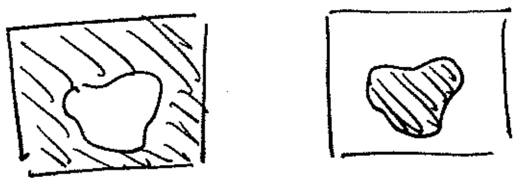
There is another aspect of the expression on p. 12 that also deserves our attention. Notice that

$$\psi(\vec{x})\Big|_{\text{circular aperture}} + \psi(\vec{x})\Big|_{\text{circular disk}}$$

$$= \frac{\psi_0'}{2R} e^{2ikR}$$

the original wave. This should not be a surprise, since the sum over all of the possible Huygens' principle sources on the plane at $z=0$ should just give back the original incoming wave. The general case is called Babinet's principle of complementary screens:

Let ψ_0 be an incoming plane wave. Let ψ_1 be the diffracted wave behind a screen with a hole of arbitrary shape. Let ψ_2 be the wave behind the complementary screen, with an obstruction where the hole is in the other case, and vice versa:



then

$$\psi_1 = \psi_0 - \psi_2$$

If the incoming wave is a plane wave moving parallel to $\hat{z}$, the wave ψ_0 does not contain any components moving at an angle $\theta \neq 0$ to the z axis. Then, away from the forward direction

$$\psi_1 = -\psi_2$$

and the complementary screens have exactly the same radiation pattern.