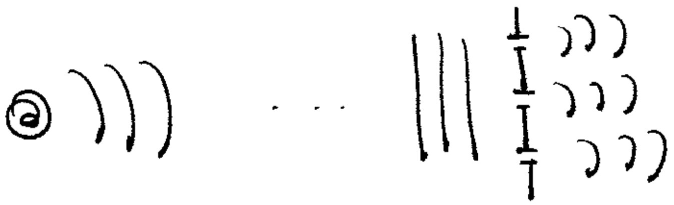
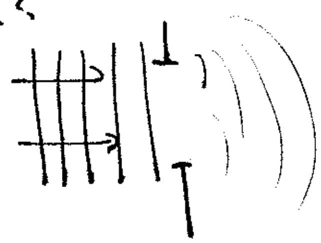
# Huygen's Principle

May 9

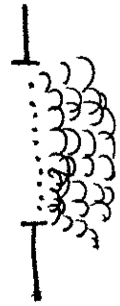
In the last lecture, we considered interference phenomena with systems of discrete sources. One of the situations we discussed was one in which a source of radiation is shielded by a screen:



The next question is, naturally, what if the screen has a large hole in it? How do we compute the radiation field to the right of the screen?

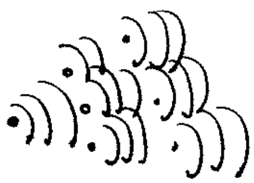


Here is a reasonable way to answer the question. Consider the whole interior of the hole as a collection of radiators.



By summing coherently over these sources, we can find the radiation field that they produce.

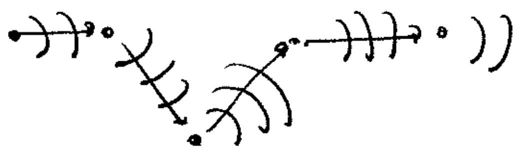
This picture is an example of a general picture of wave propagation due to Huygens (1690). Huygens argued that, as a wave moves forward, each point in the wave field can be considered as a source of additional radiation



The coherent sum of these radiated waves is the macroscopic outgoing wave.

Huygens' picture is very beautiful. Though it is difficult to make this calculation mathematically precise, it can be made precise and does give a solution of the wave equation. This formalism — the description of a wave field as a continuous ~~integral~~ over trajectories in space — had to wait for the twentieth century and the work of Richard Feynman. However, there are aspects of Huygens' picture that we can justify using less powerful mathematical methods. I would like to discuss two of these now.

Before thinking about apertures, I would like to analyze another problem, the motion of waves in a medium with a variable index of refraction. Consider a wave incident on the medium with frequency $\omega$. Then the whole wave field will oscillate at $\omega$. A typical element of the wave in Huygen's picture is



Near each point, the local wavelength is

$$\lambda(\vec{x}) = \frac{2\pi}{\omega}\frac{c}{n(\vec{x})}$$

Between two points separated by $d\vec{x}$, the number of wavelengths accumulated in the wave is

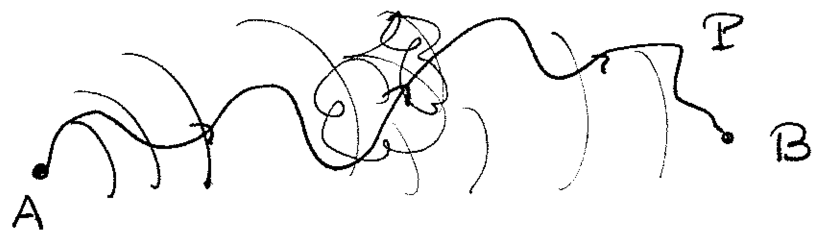
$$(\#\text{ of }\lambda\text{'s}) = ds \cdot \frac{\omega\, n(x)}{2\pi c}$$

where $ds = [d\vec{x}\cdot d\vec{x}]^{\frac{1}{2}}$ = differential of path length.

Along a complete Huygens path, the total number of wavelengths accumulated is

$$\# = \int ds\,(n(x)/c)\cdot\frac{\omega}{2\pi}$$

For a source at A and a receiver at B




The number of wavelength from A to B along the path P is

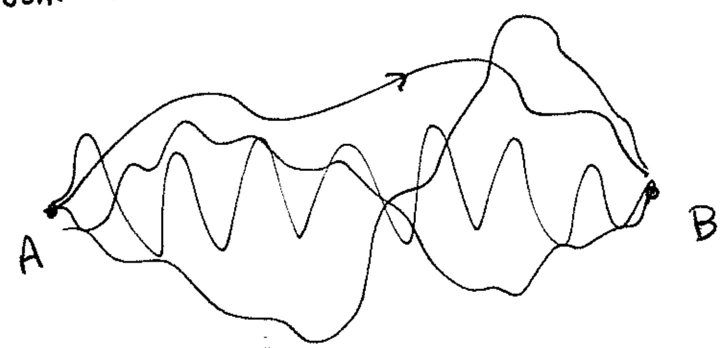
$$\# = \frac{\omega}{2\pi} \, T[P]$$

where

$$T[P] = \int_P ds \, \frac{n(x)}{c}$$

Assuming that the wave moves at speed $c/n(x)$ [Newton had this wrong in the seventeenth century, but Huygens and Fermat had it right], $T[P]$ is the time required to go along the path from A to B.

Now we have to sum the wave field coherently over all paths P from A to B.



Typically the wave travelling along P arrives at B with a random phase and cancels the wave travelling along another path P'. However, if we can find a bundle of Paths for which the time is approximately the same, the waves travelling along these paths will all arrive in phase and will sum constructively.

The condition for such constructive interference is that T[P] should be stationary with respect to a small change in the path.

$$\delta \int_A^B ds \; \frac{n(\vec{x})}{c} = 0$$

This is a variational principle, Fermat's principle of least time. From this principle, we can derive a differential equation for the path: Let $\sigma$ be a parameter of the path

$$T[\vec{x}(\sigma)] = \int d\sigma \left( \left[ \frac{d\vec{x}}{d\sigma} \right]^2 \right)^{\frac{1}{2}} \frac{n(\vec{x})}{c}$$

$$\delta T = \int d\sigma \left\{ \frac{\frac{d\vec{x}}{d\sigma} \cdot \frac{d}{d\sigma} \delta\vec{x}(\sigma)}{\left[ \left( \frac{d\vec{x}}{d\sigma} \right)^2 \right]^{\frac{1}{2}}} \frac{n(\vec{x})}{c} + \left[ \left( \frac{dx}{d\sigma} \right)^2 \right]^{\frac{1}{2}} \delta\vec{x}(\sigma) \cdot \frac{\vec{\nabla} n}{c} \right\}$$

The method for dealing with such an expression is to integrate by parts to isolate $\delta\vec{x}(\sigma)$:

$$= \int d\sigma \; \frac{\delta\vec{x}(\sigma)}{c} \cdot \left\{ -\frac{d}{d\sigma}\left( \frac{(d\vec{x}/d\sigma)\cdot n(x)}{\left[(d\vec{x}/d\sigma)^2\right]^{\frac{1}{2}}} \right) + \vec{\nabla} n \left[\left(\frac{d\vec{x}}{d\sigma}\right)^2\right]^{\frac{1}{2}} \right\}$$

This expression vanishes for any $\delta\vec{x}(\sigma)$ if

$$0 = \frac{d}{d\sigma}\left( \frac{\frac{d\vec{x}}{d\sigma}}{\left[\left(\frac{d\vec{x}}{d\sigma}\right)^2\right]^{\frac{1}{2}}} \right) \cdot n(x) + \frac{\frac{d\vec{x}}{d\sigma} \; \frac{d\vec{x}}{d\sigma}\cdot\vec{\nabla} n}{\left(\left[\frac{d\vec{x}}{d\sigma}\right]^2\right)^{\frac{1}{2}}} - \vec{\nabla} n \left[\left(\frac{d\vec{x}}{d\sigma}\right)^2\right]^{\frac{1}{2}}$$

or

$$\frac{d}{d\sigma}\left( \frac{\frac{d\vec{x}}{d\sigma}}{\left[\left(\frac{d\vec{x}}{d\sigma}\right)^2\right]^{\frac{1}{2}}} \right) = \left[\left(\frac{d\vec{x}}{d\sigma}\right)^2\right]^{\frac{1}{2}} \cdot \left(\vec{\nabla} n\right)_{\perp}$$

The light path is bent by the gradient of $n$ perpendicular to the trajectory.

If $n$ is constant

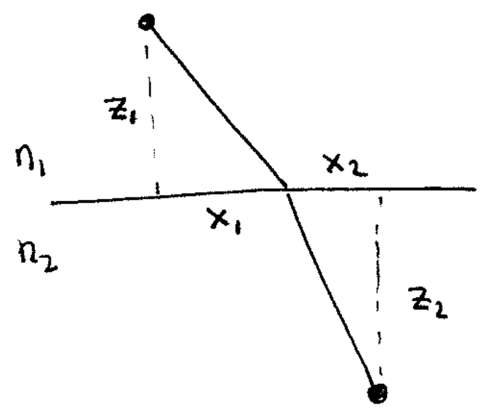
$$\frac{d}{d\sigma}\left( \frac{(d\vec{x}/d\sigma)}{\left[(d\vec{x}/d\sigma)^2\right]^{\frac{1}{2}}} \right) = 0 \quad \text{or} \quad \frac{d\vec{x}}{d\sigma} = (\text{constant})$$

Taking $\sigma = z$, the solution is

$$\vec{x} = (Az + x_0, \; Bz + y_0, \; z)$$

that is the light path is a straight line.

What if the light is propagating from a point in a medium of index of refraction $n_1$ to a point in another medium with index of refraction $n_2$. In each medium the light obviously travels in a straight line. So a typical path looks like:

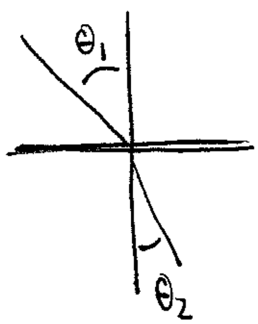



If $x_1 + x_2 = X$, we can vary over $x_1$ and find the trajectory of least time:

$$T = \left[x_1^2 + z_1^2\right]^{\frac{1}{2}} \frac{n_1}{c} + \left[(X - x_1)^2 + z_2^2\right]^{\frac{1}{2}} \frac{n_2}{c}$$

$$\frac{\partial T}{\partial x_1} = 0 = \frac{x_1}{\left[x_1^2 + z_1^2\right]^{\frac{1}{2}}} \frac{n_1}{c} - \frac{(X - x_1)}{\left[(X - x_1)^2 + z_2^2\right]^{\frac{1}{2}}} \frac{n_2}{c}$$

Now :



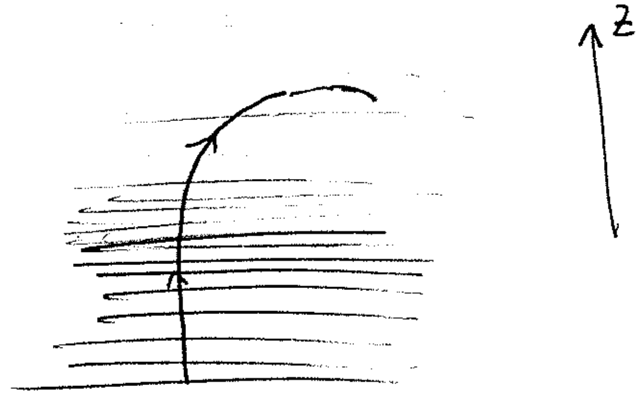

$$\frac{x_1}{\left[x_1^2 + z_1^2\right]^{\frac{1}{2}}} = \sin\theta_1 \qquad \frac{X - x_1}{\left[(X - x_1)^2 + z_2^2\right]^{\frac{1}{2}}} = \sin\theta_2$$

so the condition from the principle of least time is

$$n_1 \sin\Theta_1 = n_2 \sin\Theta_2$$

which is exactly Snell's law! If we approximate an arbitrary medium by finite elements with constant index of refraction, we can go backwards from Snell's law to derive the principle of least time.

Here is a more involved example of this type. Consider a medium for which $n(\vec{x})$ decreases linearly with height $z$. By Snell's law, an upward going light ray bends outward. Eventually, it meets a condition for total internal reflection and turns around.




Let's see if the solution to our differential equation has this property. Write

$$n(x) = n_0 (1 - \alpha z)$$

and $\vec{x}(\sigma) = (x(z), z)$ for $\sigma = z$

The differential equation on p. 6 is:

$\hat{x}$-component:
$$\frac{d}{dz}\left(\frac{dx/dz}{\left[\left(\frac{dx}{dz}\right)^2+1\right]^{1/2}}\right) n(z) + \frac{\frac{dx}{dz}}{\left[\left(\frac{dx}{dz}\right)^2+1\right]^{1/2}} \frac{dn}{dz} = 0$$

or
$$\frac{d}{dz}\left(\frac{n(z)\,\frac{dx}{dz}}{\left[\left(\frac{dx}{dz}\right)^2+1\right]^{1/2}}\right) = 0$$

or
$$\frac{n(z)\,\frac{dx}{dz}}{\left[\left(\frac{dx}{dz}\right)^2+1\right]^{1/2}} = C$$

$$\frac{dx}{dz} = \frac{C}{\left[n^2(z) - C^2\right]^{1/2}}$$

putting $n = n_0(1-\alpha z)$ $\qquad C = n_0 c$

$$\frac{dx}{dz} = \frac{c}{\left[(1-\alpha z)^2 - c^2\right]^{1/2}}$$

or
$$x = \frac{c}{\alpha}\cosh^{-1}\left(\frac{1-\alpha z}{c}\right)$$

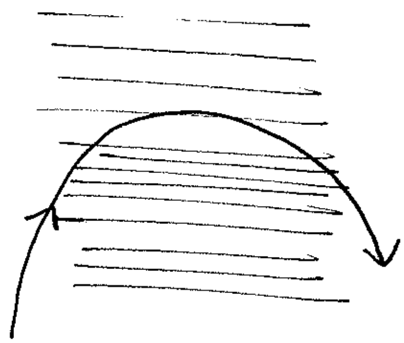
so that

$$z = \frac{1}{\alpha}\left(1 - c \cosh \frac{\alpha x}{c}\right)$$

this is a hyperbolic trajectory that turns around at

$$z = \frac{1}{\alpha}(1-c)$$

i.e. at the point where $(1-\alpha z)^2 = c^2$



It is remarkable how, in this analysis, we used a wave picture to derive a variational principle for a light ray trajectory. It is interesting to ask whether the variational principle of Lagrangian mechanics

$$\delta S = \delta \int dt\, L[x, \dot{x}] = 0$$

arises in the same way. Of course, this is exactly what happens if we base classical mechanics on quantum mechanics, where particles become waves.