Interference and Coherence

May 7

In our discussion of antennas, we found that extended sources, or multiple sources of radiation spaced out at distances comparable to the wave length of the radiation, produce complex and interesting radiation patterns. In the next few lectures, will will study this systematically, first for point sources of radiation, then for extended sources such as apertures of size much greater than a wavelength of light.

My whole discussion of this problem will be given in the scalar wave theory only. To a first approximation, we can represent the production of vector light waves from an extended source by attaching a polarization vector to the scalar wave. The cases where this approximation is inadequate are much more complicated and lie beyond the scope of this course.

In this lecture, I would like to discuss interference patterns of point sources. The point source of scalar radiation emits a spherical wave:

$$\phi(t,\vec{x}) = \mathrm{Re}\left\{ \Phi_0 \frac{e^{-i\omega t} e^{ikr}}{4\pi r} \right\}$$

this was a component of the radiated electromagnetic waves that we studied in the previous few lectures. For a scalar wave

$$\vec{j}_\mathcal{E} = -\kappa \frac{\partial \phi}{\partial t} \vec{\nabla}\phi$$

where κ is some constant. Then, for a spherical wave

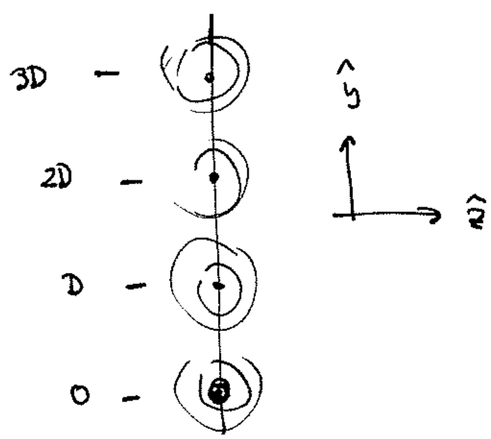
$$\langle \vec{j}_\mathcal{E} \rangle = \kappa \omega k \, \hat{r} \cdot \frac{1}{2} \cdot \frac{1}{r^2}$$

The integral over an element of solid angle is

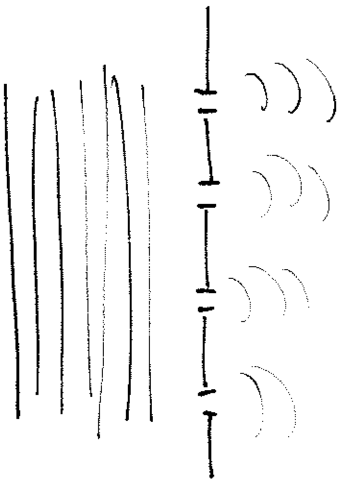
$$\int d\Omega \, \hat{n} \langle \vec{j}_\mathcal{E} \rangle \cdot r^2 = \int d\Omega \; \frac{\kappa \omega k}{2}$$

independent of r. So, energy flow out to infinity isotropically in this spherical wave.

Now consider a collection of sources spaced out along the $\hat{y}$ axis with separation D.



Equally well, we could think about a screen illuminated by a plane wave with N small holes at the spacing D:



To find the wave field to the right of the sources in either case, we superpose the spherical waves:

$$\phi = \mathrm{Re}\, \phi_0 e^{-i\omega t} \sum_{j=0}^{N-1} \frac{e^{ik|\vec{r} - \hat{y} j D|}}{4\pi |\vec{r} - \hat{y} j D|}$$

In the denominator, we can approximate

$$\frac{1}{|r - \hat{y} j D|} \approx \frac{1}{r}$$

as long as $|\vec{r}| \gg ND$ = (size of array)

In the exponent, the expansion is more subtle:

$$|\vec{r} - \hat{y} j D| = (r^2 - 2\vec{r}\cdot\hat{y}\, jD + (jD)^2)^{1/2}$$

$$= r - \frac{\vec{r}}{r}\cdot\hat{y}\, jD + \frac{1}{2}\frac{(jD)^2}{r} - \frac{1}{8}\frac{(2\vec{r}\cdot\hat{y}\, jD + \cdots)^2}{r^3}$$

$$k|\vec{r} - \hat{y} j D| = kr - k\hat{r}\cdot\hat{y}\, jD + \frac{1}{2}\frac{k}{r}\left[(jD)^2 - (\hat{r}\cdot\hat{y}\, jD)^2\right] + \cdots$$

The third term is of order

$$\frac{k}{r}(ND)^2 \cong \frac{ND}{\lambda}\cdot\frac{ND}{r}$$

so this term is negligible only for sufficiently large r that

$$r \gg \frac{(ND)^2}{\lambda} = \frac{(\text{source size})^2}{\lambda} \quad \left(\begin{array}{c}\text{Fraunhofer}\\ \underline{\text{approximation}}\end{array}\right)$$

If the source is much larger than λ, this is a much more severe criterion than $r \gg$ (source size), but nevertheless it can often be satisfied. I will use this approximation in the rest of the lecture.

Making the Fraunhofer approximation, we see that the wave field reduces to

$$\phi = \mathrm{Re}\ \phi_0\, e^{-i\omega t}\,\frac{e^{ikr}}{4\pi r}\sum_{j=0}^{N-1} e^{-ij(kD\,\hat{r}\cdot\hat{y})}$$

We recognize the function

$$f(\alpha) = \sum_{j=0}^{N-1} e^{-ij\alpha}$$

that we found in our discussion of antenna arrays.

It should now be clear that this function arises in the radiation pattern of any array of N radiators arranged along a line. In view of its importance, I would like to go over the properties of this function with more care.

As we saw before

$$f(\alpha) = \sum_{j=0}^{N-1} e^{-ij\alpha}$$

$$= 1 + e^{-i\alpha} + e^{-2i\alpha} + \cdots + e^{-i(N-1)\alpha}$$

$$= \frac{1 - e^{-iN\alpha}}{1 - e^{-i\alpha}}$$

so, up to an overall phase

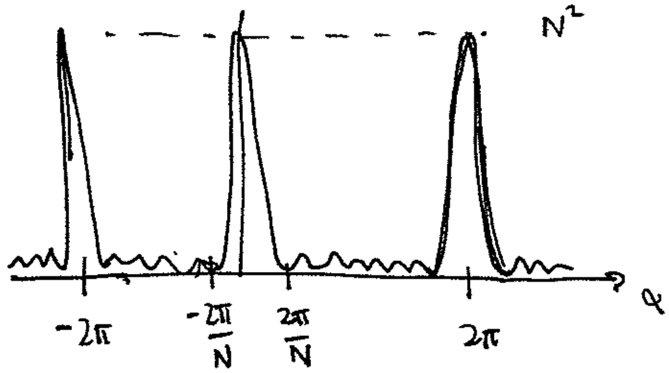
$$|f(\alpha)|^2 = \left| \frac{\sin N\alpha/2}{\sin \alpha/2} \right|^2$$

This function is periodic with period $\alpha \to \alpha + 2\pi$. It has a maximum at $\alpha = 0$, where

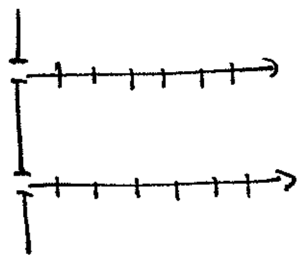
$$|f(\alpha=0)|^2 = |N|^2$$

and identical maxima at $\alpha = 2\pi m$. The function has zeros at $\alpha = \frac{2\pi k}{N}$ when k is not a multiple of N. Between these zeros are subsidiary maxima where

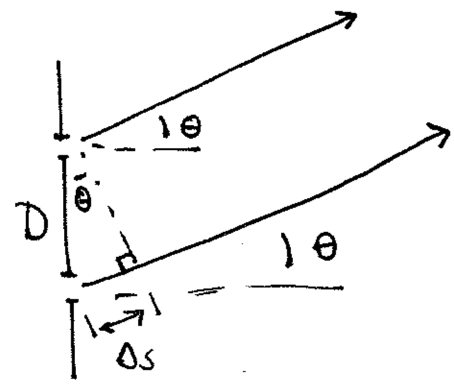
$|f(\alpha)|^2$ takes values of order 1. Then $|f(\alpha)|^2$ has the overall form:



How do we explain the large values of $|f(\alpha)|^2$ at its maxima? These can be seen geometrically, beginning with the case of two slits or two radiators. Radiation going forward to a distant point has the same path length from either slit:



However, if the radiation goes off at an angle, there is a difference in the path lengths:



$\Delta s = D \sin\theta$

so the lower wave lags by

$$\frac{\Delta s}{\lambda} = \frac{D}{\lambda} \sin\theta \quad \text{wavelengths}$$

or

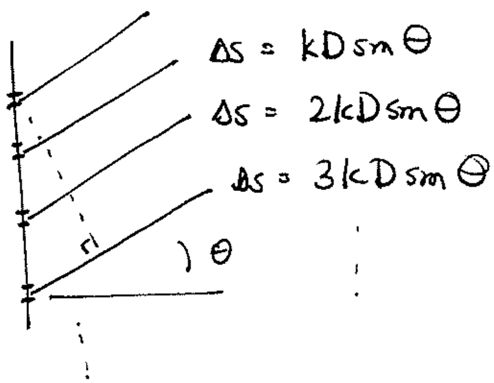
$$2\pi \frac{\Delta s}{\lambda} = kD \sin\theta \quad \text{radians.}$$

more generally, the delay in radians is

$$kD\, \hat{r}\cdot\hat{y} = \text{argument of } f(\alpha)$$

so $kD\,\hat{r}\cdot\hat{y}$ or $kD\sin\theta = 2\pi n$ is the criterion that the crests arrive in phase. For two sources, this leads to double the amplitude or 4× the intensity. For

N sources

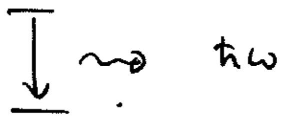


the criterion $kD\sin\theta = 2\pi n$ ensures that all N waves arrive in phase, giving

$$\text{wave amplitude} \sim N$$

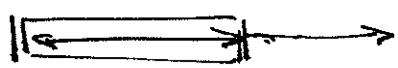
$$\text{intensity} \sim N^2.$$

When the system of sources becomes large, we should ask, how large can ΔS be in order for us to be sure that wave crests separated by ΔS are still arriving in phase. This is a question about the time or distance over which a light wave remains coherent. To answer this question, we have to look into how the light is produced. For visible light, this is quantum mechanical process in which photons are generated as individual atoms fall down from excited states into their ground states:



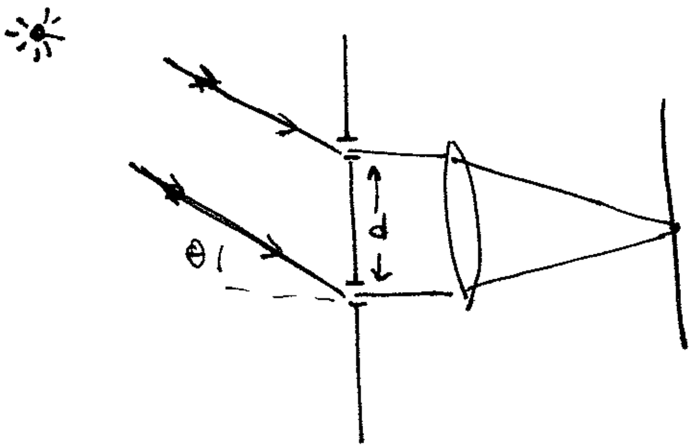
In a lamp or discharge tube, the excited atoms are prepared by heating, and the de-excitation events have no relation to one another. A typical photon emission event takes $\sim 10^{-9}$ sec, leading to a wave that is coherent over $c \cdot (10^{-9}\,\text{sec}) \sim 10\,\text{cm}$. Over much larger distances, the components of a light beam are incoherent and do not show interference effects.

We can do much better if the light is emitted by a laser. A laser is a medium in which an electromagnetic wave stimulates atoms to emit photons in phase with the wave. If the laser medium is placed between two mirrors in a cavity



the coherence length will be (length of the cavity) $\cdot$ (# of reflections), which may be a distance of km! Occasionally, this long coherence length is valuable. For example, the LIGO gravitational wave observatory is attempting to use interferometers with arms of length <u>2km</u> to measure changes in the metric of space induced by gravity waves passing by the earth.

I would now like to discuss some applications of multi-slit interference patterns, beginning with the simplest case of a two-slit pattern. Consider first a point source of light illuminating a two-slit aperture:



at angles $\sin\theta = m\frac{\lambda}{d}$ (or $\theta = m\frac{\lambda}{d}$ for small angles), the two-source pattern has a maximum. For

$$\theta \doteq (m+\tfrac{1}{2})\frac{\lambda}{d}$$

the two-source pattern has a zero.

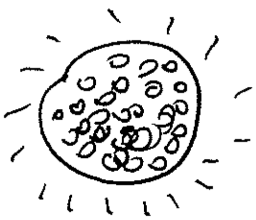
Now consider what happens if there are two sources with angular separation $\Delta\theta$. As long as $\Delta\theta \ll \frac{\lambda}{d}$, the two sources cannot be resolved and we see a sharp 2-slit interference pattern. Now decrease d until

$$\Delta\theta = \frac{1}{2}\frac{\lambda}{d} \qquad \text{i.e.} \qquad d = \frac{\lambda}{2\Delta\theta}$$

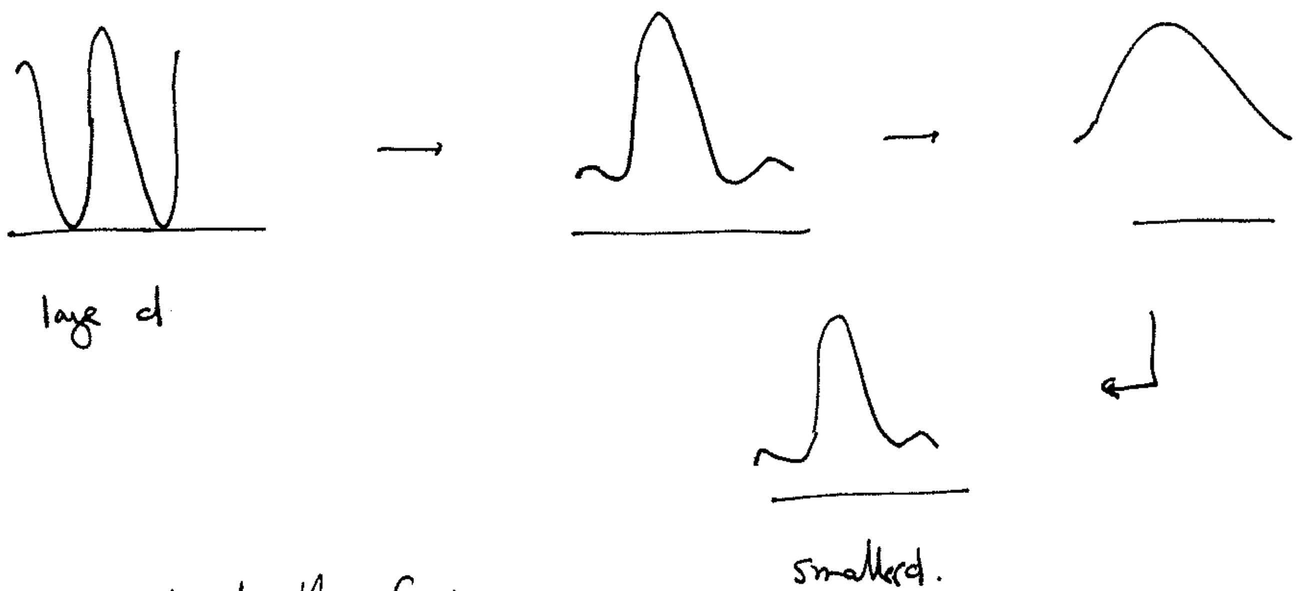
now the maximum of the pattern from one source falls on the minimum of the pattern from the other source, and we see no sharp maximum or minimum, only a blur.

Fizeau and Michelson actually used this method to determine the separation of stars in double-star systems. The angular separation of the two stars is determined by the slit separation d at which the first minimum of the interference pattern disappears.

It is possible to generalize this technique to actually measure stellar sizes. A star can be modelled as a disk in which every point radiates incoherently



For $d \gg \frac{\lambda}{\alpha}$, α = angular diameter, the star gives a clean 2-slit diffraction pattern. As d increases, though, the maxima from some regions overlap the minima from other regions of the disk



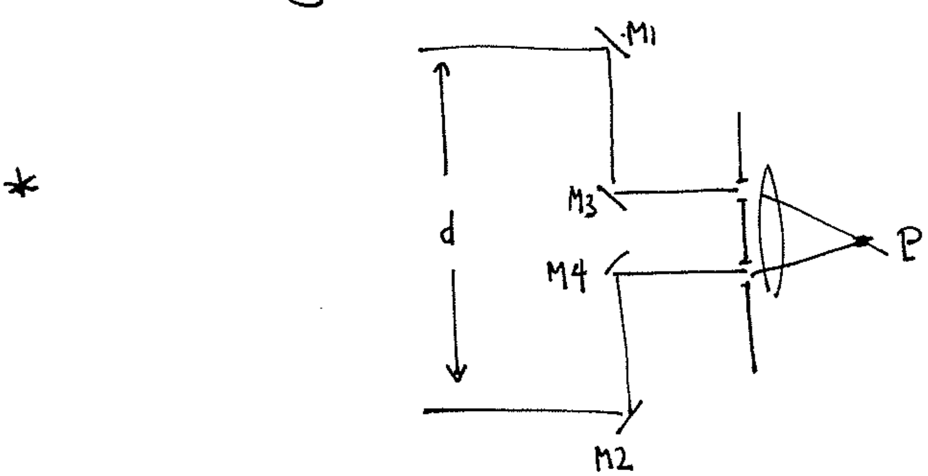
The point at which the first minimum is least clear is given by

$$d = A \frac{\lambda}{\alpha}$$

where we have seen ~~that~~ $A = \frac{1}{2}$ for 2 sources separated by α

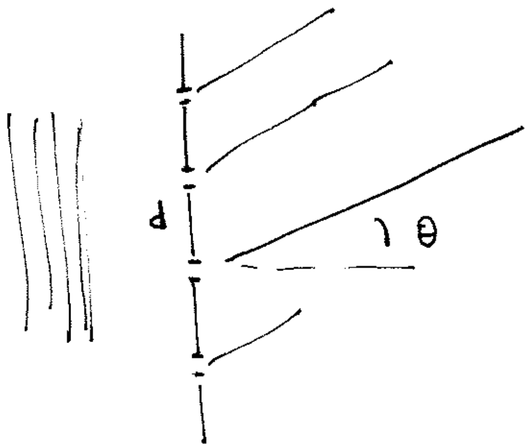
and $A = 1.22$ for a disk of angular diameter α.

Since d must become larger as α becomes smaller, Michelson used the following device to measure stellar diameters:



d is varied in such a way as to keep the optical path M1-M3-P equal to the path M2-M4-P.

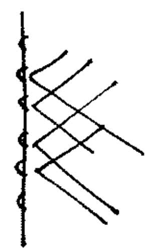
Now let's discuss a few examples of the N-source pattern. At forward angles where $\sin\theta \simeq 0$



$$\text{Intensity} \propto \left| \frac{\sin Nkd\theta/2}{\sin kd\theta/2} \right|^2$$

We can also set up this pattern by making a set of scratches on a plate that reflect light at all angles:

a "diffraction grating"



The angular width of a beam is $\Delta\theta = \frac{2\pi}{Nkd} = \frac{\lambda}{Nd}$

In a practical example, consider a diffraction grating ruled at $d = 0.1$ mm spacing over a distance of 10 cm. Then $N = 10^3$

$$\Delta\theta = \frac{\lambda}{Nd} \cong \frac{5000\,\text{Å}}{10\,\text{cm}} = 5\times10^{-6} \text{ radians.}$$

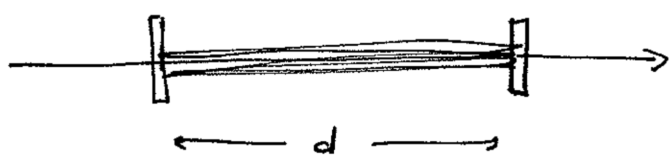
green light

The resolution in λ is

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\theta}{\theta} = \frac{\lambda/Nd}{\lambda/d} = \frac{1}{N}$$

so this grating will resolve $\frac{\Delta\lambda}{\lambda} \sim 10^{-3}$.

The N-source interference pattern can be set up not only in space but also in time. Consider, for example, the Fabry-Perot interferometer. This is a gap of length d between two partially silvered mirrors:



~~[illegible]~~

Let the reflection coefficient be R, with $R \approx 1$.
Let the optical path for one back and forth traversal be 2d. Then the wave that emerges at the

far end is

$$(1-R)\,(e^{ikd})\,(1 + R\,e^{2ikd} + R^2 e^{4ikd} + \cdots)$$

$$= (1-R)\,e^{ikd}\,\frac{1}{1-Re^{2ikd}}$$

write $1-R = T$. The intensity is then

$$I = \frac{T^2}{\left|\,1-e^{2ikd} + T e^{2ikd}\right|^2}$$

Near the maxima where $e^{2ikd} \approx 1$ $\quad 2ikd \approx 2\pi n + (\text{small})$

$$I = \frac{|T|^2}{4\sin^2 kd + |T|^2}$$

If T is small, this function is sharply peaked about values of d or λ satisfying the condition

$$kd = n\pi$$

or

$$\frac{2d}{\lambda} = n$$