Physics 122

Intermediate Electricity and Magnetism

Syllabus:

Relativity, electrodynamics, radiation

Radiation from accelerating particles

Antennas and other radiating systems

Scattering of electromagnetic waves

Diffraction

References

textbook:

Griffiths, Introduction to Electrodynamics

also

Heald and Marion, Classical Electromagnetic Radiation

Feynman Leighton and Sands, The Feynman Lectures on Physics, vol. 2

Relativity and Electrodynamics

April 4

The main theme of Physics 122 is the interaction of electromagnetic radiation with matter. In 120, we studied static electric and magnetic fields, leading up to a derivation of Maxwell's equations. In 121, we studied electromagnetic waves and their propagation in various media. Now we need to study how electromagnetic waves are produced, and how they influence and are influenced by particles of matter.

At the end of 121, we found that the structure of electrodynamics really comes from the theory of relativity. The wave equation

$$\left(\frac{1}{c}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = \Box\phi = 0$$

turned out to be the unique linear relativistic equation for a scalar field, and Maxwell's equations turned out to be the simplest relativistic equation for a field whose source is a conserved current. Although we did not really need relativity to understand the properties of electromagnetic waves, it can make the picture of waves simpler. When we discuss the more complicated problem of the production of electromagnetic waves, it will be very useful to incorporate relativity from the start. For this reason, I will begin Physics 122 with a more careful treatment of the implications of special relativity for electromagnetism. We will develop a beautiful and general theory of the production of radiation, which we can then generalize to a variety of problems. As a bonus, we will find another, still more satisfying, derivation of Maxwell's equations.

To begin the course, I would like to recapitulate the major aspects of the theory of relativity that we discussed at the end of 121. Then we will develop some of these points further for specific application to electrodynamics.

(a) The postulates of the theory of relativity are

1. The laws of physics are the same in all inertial frames. A motion that solves their equations with respect to one inertial observer solves the equations with respect to all inertial observers.
2. The speed of light is a constant value, $c = 3.0 \times 10^8$ m/sec, with respect to all inertial observers.

(b) To implement these requirements, we considered the coordinates measured by inertial observers to be connected by <u>Lorentz transformations</u>, linear transformations that preserve the interval

$$s^2 = (ct)^2 - (\vec{x})^2$$

Let $x^\mu = (ct, \vec{x})$ $\quad x'^\mu = (ct', \vec{x}')$ be coordinates measured in two different inertial frames. Then

$$s^2 = (ct)^2 - (\vec{x})^2 = s'^2 = (ct')^2 - (\vec{x}')^2$$

is guaranteed by the relation

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu$$

Where the 4x4 matrix $\Lambda^\mu{}_\nu$ is a <u>Lorentz</u> transformation. Any continuous Lorentz transformation can be written as a product of ro<u>tations</u> and <u>boosts</u>. Examples of these are:

x' is rotated with respect to x by $\phi\hat{3}$:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\phi & \sin\phi & \\ & -\sin\phi & \cos\phi & \\ & & & 1 \end{pmatrix}$$

x' is boosted with respect to x by $v\hat{3}$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & & & \sinh\eta \\ & 1 & & \\ & & 1 & \\ \sinh\eta & & & \cosh\eta \end{pmatrix}$$

where $v/c = \tanh\eta$. Often, we write
$\cosh\eta = \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ $\sinh\eta = \beta\gamma = \frac{v/c}{\sqrt{1-v^2/c^2}}$

(c) Actually, the matrices Λ solve a more general problem:

If $\mu, \nu = 0123$

$$A^\mu = \Lambda^\mu{}_\nu A'^\nu \quad \text{and} \quad B^\mu = \Lambda^\mu{}_\nu B'^\nu$$

then the Λ are the most general matrices for which

$$A^0 B^0 - \vec{A}\cdot\vec{B} = A'^0 B'^0 - \vec{A}'\cdot\vec{B}'$$

We can define a <u>metric</u>

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Then the Λ preserve this metric:

$$\eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\sigma = \eta_{\lambda\sigma}$$

The relation

$$A^\mu = \Lambda^\mu{}_\nu A'^\nu$$

can describe the relation between the values of 4 physical quantities (A^0, A^1, A^2, A^3) measured in the frame of (x) and the analogous quantities measured in the frame of (x'). A set of 4 quantities which have this relation between their values in any two inertial frames is called a <u>covariant</u> <u>4-vector</u>.

The dual to a covariant 4-vector is a 4-component object C_μ which transforms between frames in such a way that

$$A^\mu C_\mu = A^0 C_0 + \vec{A}\cdot\vec{C}$$

is invariant. Such an object is called a contravariant 4-vector. If C_μ is a contravariant 4-vector

$$C^\mu = \eta^{\mu\nu} C_\nu$$

is a covariant 4-vector; if A^μ is a covariant 4-vector,

$$A_\mu = \eta_{\mu\nu} A^\nu$$

is contravariant.

(d) Many ~~interesting~~ physical quantities are components of 4-vectors. Some important examples are:

energy and momentum: $\quad P^\mu = \left(\frac{E}{c}, \vec{P}\right)$

charge density and current: $\quad J^\mu = (\rho c, \vec{J})$

The energy and momentum of a point particle are described by the formula:

$$P^\mu = m\frac{dx^\mu}{d\tau}$$

where τ is proper time, time measured by a clock moving with the particle. More explicitly,

$$\frac{E}{c} = \frac{mc}{\sqrt{1-v^2/c^2}} \qquad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

It is worth noting that this formula assigns an energy to a particle at rest:

$$E\Big|_{\text{rest}} = mc^2$$

We are always free to define kinetic energy as

$$T = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

If the number of each particle species is conserved, $\sum m_i c^2$ will be conserved and so T will be conserved. However, if particles can be created and destroyed, T is not obviously conserved, and there is a problem with it: Since T does not have a simple transformation law, its conservation may depend on the frame of the observer. Only the full E combines with $\vec{p}$ to give a relation:

(conservation of $E, \vec{p}$ in one frame) $\longrightarrow$ (conservation of $E, \vec{p}$ in all frames)

(e) The gradient in space-time:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$$

transforms as a contravariant 4-vector. Thus,

e.g.

① $\partial_\mu J^\mu$ is an invariant if $\partial_\mu J^\mu = 0$ in one frame, this relation is true in all frames

[and recall: $\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J}$]

② $$\partial_\mu \eta^{\mu\nu} \partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box$$

is an invariant operator. $\Box\phi = 0$ in one frame implies that this relation holds in all frames.

(f) Force $\vec{F} = \frac{d\vec{p}}{dt}$ does not have a simple transformation law from one frame to another. However, the following special cases are simple. Let $\vec{F}'$ be the force on a particle in its rest frame

$$\frac{d\vec{p}'}{dt'} = \vec{F}' \quad \text{for} \quad \vec{p}' = 0$$

Then, if $\vec{F} = \frac{d\vec{p}}{dt}$ is the force in a frame where the particle is moving with velocity $\vec{v}$:

$$\vec{F} = \vec{F}' \qquad \vec{F}' \parallel \vec{v}$$

$$\vec{F} = \frac{1}{\gamma}\vec{F}' \qquad \vec{F}' \perp \vec{v}$$

(9) The electric and magnetic fields must also be assigned simple transformation laws under Lorentz transformations. There are 6 components of electromagnetic field, and these fit into a 2-index antisymmetric tensor $F^{\mu\nu}$

$i,j = 1,2,3$

$$E^i = F^{i0} \qquad F^{ij} = -c\,\epsilon^{ijk}B^k$$

then the covariant equation

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0} J^\nu$$

contains the Maxwell equations

$\nu = 0$: $$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$$

$\nu = i$: $$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

notice that this equation requires that the current be conserved:

$$\partial_\nu (\partial_\mu F^{\mu\nu}) = 0 = \frac{1}{c\epsilon_0} \partial_\nu J^\nu$$

because F is antisymmetric

the covariant equation

$$\varepsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0$$

contains the Maxwell equations

$\mu = 0$ $\quad\quad \vec{\nabla} \cdot \vec{B} = 0$

$\mu = i$ $\quad\quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

These assignments dictate how the $\vec{E}$ and $\vec{B}$ fields transform from one frame to another. Let $\vec{E}'$, $\vec{B}'$ be the electromagnetic fields in the (x') frame. Then, in the (x) frame moving with velocity $v\hat{z}$ with respect to (x'), we will observe the fields

$$F'^{\mu\nu} = \Lambda^\mu{}_\lambda \Lambda^\nu{}_\sigma F'^{\lambda\sigma}$$

Using

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

we can write this out explicitly

$$E^1 = F^{10} = \gamma F'^{10} + \beta\gamma F'^{13}$$

$$= \gamma\left(E'^1 + \frac{v}{c}\cdot c B'^2\right)$$

$$E^2 = F^{20} = \gamma F'^{20} + \beta\gamma F'^{23}$$

$$= \gamma\left(E'^2 - \beta c B'^1\right)$$

$$E^3 = F^{30} = \gamma^2\left(F'^{30} + \beta \underbrace{F'^{33}}_{0} + \beta \underbrace{F'^{00}}_{0} + \beta^2 F'^{30}\right)$$

$$= \gamma^2(1-\beta^2) E'^3 = E'^3$$

$$B^1 = -\frac{1}{c} F^{23} = -\frac{1}{c}\gamma\left(F'^{23} + \beta F'^{20}\right)$$

$$= \gamma B'^1 - \frac{1}{c}\gamma\beta E'^2$$

$$B^2 = -\frac{1}{c} F^{31} = -\frac{1}{c}\gamma\left(F'^{31} + \beta F'^{01}\right)$$

$$= \gamma B'^2 + \frac{1}{c}\gamma\beta E'^1$$

$$B^3 = -\frac{1}{c} F^{12} = -\frac{1}{c} F'^{12} = B'^3$$

The generalization to arbitrary $\vec{V}$ is

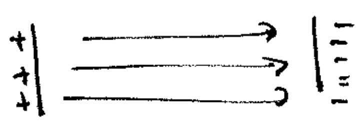
components $\perp$ to $\vec{V}$

$$\begin{cases} \vec{E} = \gamma\left(\vec{E}' - \vec{V}\times\vec{B}'\right) \\ \vec{B} = \gamma\left(\vec{B}' + \dfrac{\vec{V}}{c^2}\times\vec{E}'\right) \end{cases}$$

components $\parallel$ to $\vec{V}$

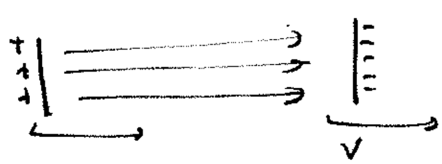
$$\begin{cases} \vec{E}_{\parallel} = \vec{E}'_{\parallel} \\ \vec{B}_{\parallel} = \vec{B}'_{\parallel} \end{cases}$$

We saw last term ~~that~~ $E_{\parallel} = E_{\parallel}$; this is what we would expect by applying Gauss' law to a capacitor at rest



$$E' = \frac{\rho}{\epsilon_0}$$

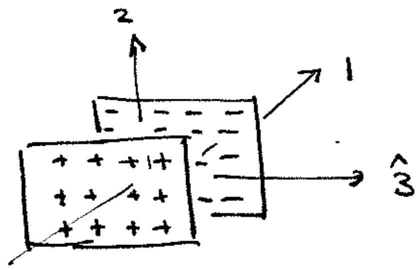
and to a capacitor in motion



$$E = \frac{\rho}{\epsilon_0}$$

since ρ C/m^2 is the same in the two cases.

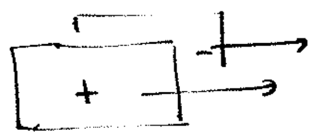
To check the $\perp$ components, consider a capacitor oriented perpendicular to the z axis:



In the frame where the capacitor is at rest, we have

$$\vec{B}' = 0 \qquad \vec{E}' = \frac{\rho'}{\epsilon_0}\,\hat{1}$$

Now boost the capacitor to velocity $v\,\hat{3}$



In the new frame, we have

$$\rho = \frac{\rho'}{\sqrt{1-v^2/c^2}} \qquad C/m^2$$

$$\vec{J} = \frac{\rho' v}{\sqrt{1-v^2/c^2}}\,\hat{3} \qquad \begin{array}{c}\text{surface current}\\ A/m\end{array}$$

The surface chge gives rise to $E' = \frac{\rho}{\epsilon_0} = \gamma E''$

The surface current gives rise to

$$B^2 = \mu_0 j = \mu_0 V \epsilon_0 E^1$$

$$= \frac{1}{c^2} V \gamma E'^1$$

indeed!

$$E^1 = \gamma E'^1$$

$$B^2 = \gamma \frac{V}{c^2} E'^1$$

as we predicted

The transformation laws of electric and magnetic fields have the interesting property that certain combinations remain invariant to a change of frame. Notice that

$$E^2 - c^2 B^2 =$$

$$\gamma^2 (\vec{E}'_\perp - \vec{V} \times \vec{B}_\perp)^2 + (E'_\parallel)^2 - c^2 \gamma^2 (B'_\perp + \frac{\vec{V}}{c^2} \times \vec{E}_\perp)^2 - c^2 B'^2_\parallel$$

$$= \gamma^2 (E'_\perp)^2 - \cancel{2\gamma^2 \vec{E}'_\perp \cdot (\vec{V} \times \vec{B}'_\perp)} + \gamma^2 (\vec{V} \times \vec{B}_\perp)^2$$

$$- c^2 \gamma^2 (B'_\perp)^2 - \cancel{2\gamma^2 \vec{B}'_\perp \cdot (\vec{V} \times \vec{E}'_\perp)} - \frac{\gamma^2}{c^2} (\vec{V} \times \vec{E}_\perp)^2$$

$$+ (E'_\parallel)^2 - c^2 (B'_\parallel)^2$$

$$= \gamma^2 (1 - V^2/c^2)(E'_\perp)^2 - \gamma^2 c^2 (1 - V^2/c^2)(B'_\perp)^2$$

$$+ (E'_\parallel)^2 - c^2 (B'_\parallel)^2$$

$$= E'^2 - c^2 B'^2$$

and

$$\vec{E}\cdot\vec{B} = E'_{\parallel}\cdot B'_{\parallel} + \gamma(\vec{E}'_{\perp} - \vec{v}\times\vec{B}'_{\perp})\cdot\gamma(\vec{B}'_{\perp} + \frac{\vec{v}}{c^2}\times\vec{E}'_{\perp})$$

$$= E'_{\parallel}\cdot B'_{\parallel} + \gamma^2\left[E'_{\perp}\cdot B'_{\perp} - \frac{1}{c^2}(\vec{v}\times\vec{B}'_{\perp})\cdot(\vec{v}\times\vec{E}'_{\perp})\right]$$

$$- \gamma\,\underbrace{(\vec{v}\times\vec{B}'_{\perp})\cdot\vec{B}'_{\perp}}_{0} + \gamma\,\underbrace{\vec{E}'_{\perp}\cdot(\frac{\vec{v}}{c^2}\times\vec{E}'_{\perp})}_{0}$$

$$= E'_{\parallel}\cdot B'_{\parallel} + \gamma^2(1 - v^2/c^2)\,E'_{\perp}\cdot B'_{\perp}$$

$$= \vec{E}'\cdot\vec{B}'$$

These invariant combinations are easy to understand in 4-vector notation. The invariant

$$\frac{1}{2}(F^{\mu\nu}F_{\mu\nu})$$

$$= F^{i0}F_{i0} + \frac{1}{2}F^{ij}F_{ij}$$

$$= -F^{i0}F^{i0} + \frac{1}{2}F^{ij}F^{ij}$$

$$= -E^2 + c^2(\epsilon^{ijk}B^k)^2$$

$$(-\frac{1}{2}F^{\mu\nu}F_{\mu\nu}) = E^2 - c^2B^2$$

Similarly,

$$\frac{1}{8}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} = \frac{1}{2}\epsilon^{0ijk}F_{0i}F_{jk}$$

$$= \frac{1}{2}\epsilon^{ijk}F^{i0}F^{jk}$$

$$= \frac{1}{2}\epsilon^{ijk}E^{i}(-c\epsilon^{jkl}B^{k})$$

$$= -c\vec{E}\cdot\vec{B}$$

so

$$-\frac{1}{8c}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} = \vec{E}\cdot\vec{B}$$

The Lorentz invariance of these two quantities implies

① If $|\vec{E}| > |\vec{B}|$ in one frame, this is true in all frames

If $|\vec{B}| > |\vec{E}|$ in one frame, this is true in all frames

② If $\vec{E} = 0$ in one frame, then in all frames $\vec{B} \perp \vec{E}$

If $\vec{B} = 0$ in one frame, then in all frames $\vec{E} \perp \vec{B}$