Physics 121 – Problem Set # 7

(due Friday, May 31)

1. Griffiths, problem 9.28.
2. Griffiths, problem 9.29.
3. In this problem, you will study electromagnetic modes in a cylindrical pipe more explicitly.

   (a) Using the power series solution for $J_m(z)$, prove the identity

   $$\frac{d}{dz}J_m(z) = J_{m-1}(z) - \frac{m}{z}J_m(z) \tag{1}$$

   for $m > 0$, and show that $(d/dz)J_0(z) = -J_1(z)$.

   (b) Now consider electromagnetic waves in a cylindrical pipe of radius a with perfectly reflecting walls. The boundary conditions on the walls are: $E_\parallel = B_\perp = 0$. You can treat the interior of the pipe as vacuum, but the same boundary conditions will arise if the 'pipe' is an optical fiber with very large n. Find explicit formulae for the $\vec{E}$ and $\vec{B}$ fields in the lowest-frequency TM wave. Sketch the fields predicted by these formulae, and show that they have the form expected intuitively.

   (c) In the same system, find explicit formulae for the $\vec{E}$ and $\vec{B}$ fields in the lowest-frequency TE wave with $m = 0$. Sketch the fields predicted by these formulae, and show that they have the form expected intuitively.

   (d) Construct explicitly the lowest-frequency TE and TM waves with $m = 1$.

4. Consider electromagnetic waves in a cylindrical cavity with height d and radius r made of perfectly conducting material.

   (a) Show that the resonant frequencies are

   $$\omega_{lmn}^2 = c^2(\pi\ell/d)^2 + \omega_{nm}^2 \tag{2}$$

   where the ω_{nm} are the cutoff frequencies in a waveguide of the same radius and $\ell = 1, 2, ...$ Unlike the case of a rectangular cavity, the TE and TM modes are not degenerate.

   (b) For each of the four travelling waves constructed in the previous problem, construct explicitly the corresponding standing waves in this cavity. Sketch the oscillating fields.

5. In this problem, you will study electromagnetic modes in a cavity made of a material that is a good but not perfect conductor. For simplicity, set $\mu = \mu_0$, $\epsilon = \epsilon_0$ in the conductor.

   (a) For a damped harmonic oscillator

   $$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{3}$$

   with small damping, $\gamma << \omega_0$, the 'quality factor' Q is defined by

   $$Q = \frac{\omega_0}{\gamma} \tag{4}$$

   Show that $1/Q$ is the fraction of the total energy lost to damping in each cycle.

   (b) Next think about an electromagnetic wave reflecting from a good conductor. Assume that the electric currents in the conductor obey the relation $\vec{j} = \sigma\vec{E}$. For a perfect conductor and normal incidence, the solution for an electromagnetic wave is

   $$\begin{aligned} \vec{E} &= \mathrm{Re}\left[E_0\hat{x}(e^{ikz-i\omega t} - e^{-ikz-i\omega t})\right] \\ \vec{B} &= \mathrm{Re}\left[(E_0/c)\hat{y}(e^{ikz-i\omega t} + e^{-ikz-i\omega t})\right] \ . \end{aligned} \tag{5}$$

   Now consider the same problem of reflection with σ large but not infinite. Compute the reflection coefficient and the energy loss to the wall per cycle per unit area per unit time.

   (c) For the wave in eq. (5), show that, at the surface, $\left\langle(\partial\vec{E}/\partial z)^2\right\rangle = 2k^2E_0^2$, where the brackets denote the average over a cycle. Using this expression, compute the coefficient α in the relation

   $$\langle\Phi_E\rangle = \alpha\left\langle(\partial\vec{E}/\partial z)^2\right\rangle \tag{6}$$

   where $\langle\Phi_E\rangle$ is the flux of energy dissipated in the wall averaged over a cycle. Relate α to the conductivity σ, and the skin depth δ. I think it is reasonable to use this relation to compute the dissipation of energy in a conducting wall in any situation with oscillating $\vec{E}$ fields for which $\delta \ll \lambda$. Why (or why not)?

   (d) Use the approximation in (c) to compute the energy loss per cycle for the lowest frequence (TE_{011}) mode of an electromagnetic cavity formed as a cubic box with sides of length a. That is, compute the $\vec{E}$ field configuration in the box, compute $\left\langle(\partial\vec{E}/\partial z)^2\right\rangle$ on the walls, and then use eq. (6) to compute the energy loss.

   (e) Find an expression for the Q of this oscillator.

6. If you are going to use Bessel functions in practical problems, you will need to know how to generate them numerically. This problem addresses that question.

(a) For sufficiently low z, we can compute $J_m(z)$ from the power series in z:

$$J_m(z) = (\frac{1}{2}z)^m \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!(m+k)!} \tag{7}$$

For $m = 0, 1, 2$, estimate how many terms are needed to compute $J_m(z)$ to 1% accuracy for $z = 1, 2, 3$.

(b) For sufficiently high z, we can compute $J_m(z)$ from the asymptotic series applicable for large z: If $\chi = z - m\pi/2 - \pi/4$,

$$\begin{aligned} J_m(z) &= \sqrt{\frac{2}{\pi z}}\Bigg[\Bigg(1 - \frac{(4m^2-1)(4m^2-9)}{2!(8z)^2} + \cdots\Bigg)\cos\chi \\ &\quad -\Bigg(\frac{(4m^2-1)}{8z} - \frac{(4m^2-1)(4m^2-9)(4m^2-25)}{3!(8z)^3} + \cdots\Bigg)\sin\chi\Bigg] \end{aligned} \tag{8}$$

More terms in this series are available in the book of Abramowitz and Stegun. Estimate the values of z for which this 4-term series gives 1% accuracy for $m = 0, 1, 2$.

(c) For each value of m up to 2, find a value z_* such that the use of the series in (a) for $z < z_*$ and the use of the series in (b) for $z > z_*$ gives an evaluation of $J_m(z)$ with 1% accuracy uniformly.

(d) Use this idea to write a Java program to compute $J_m(z)$. The file `Bessel.java`, which you can download from the Physics 121 web site `http://www.slac.stanford.edu/~mpeskin/Physics121/java/`, contains a shell of a program `Bessel(z)`, which is intended to return the value of $J_m(z)$. Note that `m` is specified in the code. Modify this program so that, for $m = 0, 1, 2$, it returns an accurate value.

(e) Now download also the files `Bessel.html`, `BesselGUI.java`, `BesselGUI.jar`, and the new revised version of `PhysicsApplet.java`. With the modified `Bessel.java`, these files implement an applet that plots Bessel functions $J_m(kx)$ in the interval $0 < x < 1.5$ for fixed m. The value of k can be adjusted with the slidebar. When you press the button 'add', the new curve is added to the plot. Use this applet to find, by eye, the first three zeros of $J_m(z)$ for $m = 0, 1, 2$.

(f) Another strategy for computing Bessel functions uses the recursion relation

$$J_{m+1}(z) = \frac{2m}{z} J_m(z) - J_{m-1}(z) \tag{9}$$

Prove this relation using the power series in (a). Use this method as an alternative way of computing $J_2(z)$.

(g) Implement a computation of $J_3(z)$, and find the first three zeros of this function.

Hand in your code for the `Bessel` class, your solutions to (a), (b), (c), (e), (f), and (g), and illustrative plots.

```
import java.awt.*;
import java.util.*;
import java.applet.*;

public class Bessel extends BesselGUI {

    int m = 0;

    double Bessel(double z){
        double y = 1.0;
        if ( m == 0){
            y = Math.cos(z);
        } else if (m ==1 ){
            y = Math.sin(z);
        }  else if (m == 2){
            y = 0.0;
        } else {
            y = 0.0;
        }
        return y;
    }
}
```

Figure 1: The source file `Bessel.java`.