Physics 121 – Problem Set # 3

(due Friday, April 26)

1. Use the method of contour integration to do the following integrals:

$$\begin{aligned} \text{a.)} \quad & \int_{-\infty}^{\infty} dx \, \frac{1}{x^2+2x+5} \\ \text{b.)} \quad & \int_{-\infty}^{\infty} dx \, \frac{1}{(x^2+2x+5)^2} \\ \text{c.)} \quad & \int_{0}^{2\pi} d\phi \, \frac{1}{1+b^2+2b\cos\phi} \end{aligned} \tag{1}$$

   In (c), let $z = e^{i\phi}$ and take b to be real and less than 1.

2. Show that the function

$$f(z) = \frac{1}{e^{2\pi i z}-1} \tag{2}$$

   has a simple pole at $z = n$ for every integer n, with residue 1 at each point. Show also that $f(z)$ has no other singularities and is bounded as $z \to \infty$ above or below the real axis. Use these observations to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{3}$$

   by writing the left-hand side as a contour integral involving $f(z)$.

3. Analyze the integral

$$I(\mu, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{\sqrt{k^2+\mu^2}} \ . \tag{4}$$

   for $x > 0$. Show that this integral is only a function of the combination (μx). Show that, as $x \to \infty$, $I(\mu x)$ decreases exponentially as $e^{-\mu x}$. A nifty contour deformation is very helpful. Notice that the integrand does not have an isolated singularity in the complex plane but rather two lines of singularites starting from $k = \pm i\mu$. We can take these lines to run along the imaginary axis from $k = \pm i\mu$ to $k = \pm i\infty$. Show that the expression $\sqrt{k^2+\mu^2}$ is pure imaginary along these lines with the opposite sign on the left- and right-hand side. By running the contour around one of these lines, show, more exactly, that

$$I(\mu x) \sim \frac{A}{\sqrt{\mu x}} e^{-\mu x}(1 + \mathcal{O}(\frac{1}{\mu x})) \tag{5}$$

   as $x \to \infty$, and find the constant A.

4. Griffiths, problem 8.5.

5. Griffiths, problem 8.9.

6. Griffiths, problem 8.11.