Physics 121 - Midterm

Solutions

1.) a.)

$$Z = R + \left[\frac{1}{-i\omega L} + -i\omega C \right]^{-1}$$

$$= R + \left[\frac{1 - \omega^2 LC}{-i\omega L} \right]^{-1}$$

$$= R + \frac{-i\omega L}{1 - \omega^2 LC}$$

$$Z = \frac{R(1-\omega^2 LC) - i\omega L}{1-\omega^2 LC}$$

b.)

$$\frac{\tilde{V}(\omega)}{\tilde{I}(\omega)} = Z(\omega) \quad \text{so} \quad \frac{\tilde{I}(\omega)}{\tilde{V}(\omega)} = \frac{1}{Z} = \frac{1-\omega^2 LC}{R(1-\omega^2 LC) - i\omega L}$$

so $I(\omega)$ <u>vanishes</u> at $\omega = \omega_0$

c) $V(t) = V_0 \delta(t) \rightarrow \tilde{V}(\omega) = V_0$

$$\tilde{I}(\omega) = V_0 \frac{1-\omega^2 LC}{R - i\omega L - \omega^2 LCR}$$

The poles of the denominator are at ω s.t.

$$\omega^2 + i\,\frac{\omega}{RC} - \frac{1}{LC} = 0$$

$$\left(\omega + \frac{i}{2}\frac{1}{RC}\right)^2 + \frac{1}{4R^2C^2} - \frac{1}{LC} = 0$$

$$\omega = -\frac{i}{2}\frac{1}{RC} \pm \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

Inverting the Fourier transform, we find

$$e^{-i\omega t} \rightarrow e^{-t/2RC}\, e^{\pm i\bar{\omega} t}$$

so $\tau = 2RC$ $\quad\quad$ $\bar{\omega} = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} \approx \frac{1}{\sqrt{LC}}$

2.) a.) The E field and the charge stored in the capacitor decay as

$$e^{-t/RC}$$

Initially,

$$E = \frac{10^3 V}{10^{-2} m} = 10^5 \ V/m$$

$$\rho = \epsilon_0 E = 8.85 \times 10^{-12} \ C^2/Nm^2 \cdot 10^5 \ \frac{N}{C}$$

$$= 8.85 \times 10^{-7} \ C/m^2$$

The total charge stored is.

$$Q = \pi r^2 \rho = \pi (0.1m)^2 (8.85 \times 10^{-7} \ C/m^2)$$

$$= 2.78 \times 10^{-8} \ C$$

The capacitance is

$$C = \frac{Q}{V} = 2.78 \times 10^{-11} \ F$$

$$RC = 2.78 \times 10^{-8} \ sec$$

so

$$E = (10^5 \ V/m) \ e^{-t/a} \qquad a = 2.78 \times 10^{-8} sec$$

The stored energy is

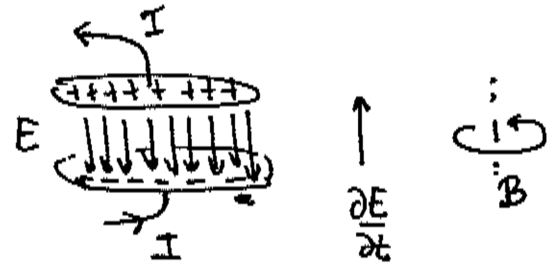
$$\frac{1}{2}\epsilon_0 E^2 \cdot \text{Vol.}$$

initially $= \frac{1}{2}\left(8.85\times10^{-12}\ C^2/Nm^2\right)\left(10^5\ N/C\right)^2 \pi\,(0.1m)^2(0.01m)$

$$= 1.39\times10^{-5}\ J$$

so $\text{Energy}(t) = 1.39\times10^{-5}\ J\ e^{-t/(a/2)}$

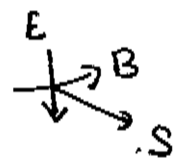
$$a/2 = 1.39\times10^{-8}\ \text{sec}$$



b) $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

$$2\pi r B = \pi r^2 \frac{1}{c^2}\left|\frac{\partial E}{\partial t}\right|$$

so $B = \frac{r}{2c^2}\left|\frac{\partial E}{\partial t}\right|$



$S = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ outward

$$= \frac{1}{\mu_0} E \cdot \frac{r}{2c^2}\frac{1}{a} E$$

$$S = \left(\frac{1}{\mu_0}\frac{r}{2c^2}\frac{1}{a}E^2\right)_{\text{initial}} e^{-t/(a/2)}$$

the coefficient is

$$\frac{1}{4\pi}\times 10^{7}\,\frac{(C/sec)^2}{N}\;\frac{r\,(m)}{2\,(3.0\times10^{8}\,m/sec)^2}\;\frac{1}{2.78\times10^{-8}\,sec}\,(10^{5}\,N/C)^2$$

$$= 1.59\times10^{6}\cdot r\,(m)\cdot J/sec\cdot m^2$$

so

$$S = 1.59\times10^{6}\cdot r\,(m)\;J/m^2 sec\cdot e^{-t/(a/2)}$$

c.) Integrate over a cylinder enclosing the capacitor

$$\frac{dE}{dt} = 2\pi r\ell\cdot S$$

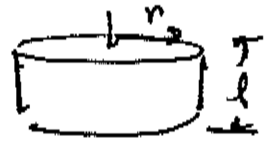
$$= 2\pi\,(0.1m)\,(0.01\,m)\cdot 1.59\times10^{6}\cdot(0.1m)\cdot e^{-t/(a/2)}$$

$$= 1.0\times10^{3}\;J/sec$$

$$\int\frac{dE}{dt}\,dt = 1.0\times10^{3}\,J/sec\cdot 1.39\times10^{-8}\,sec$$

$$= 1.39\times10^{-5}\,J \quad ✓$$

[I did not ask for this, but here is the analytical treatment :



$$C = \frac{Q}{V} = \frac{\pi r^2 \rho}{V} =$$

$$= \frac{\pi r^2 \rho}{(\rho/\epsilon_0)\, l} = \frac{\epsilon_0 \pi r^2}{l}$$

so $$a = RC = \frac{\epsilon_0 \pi r^2}{l} R$$

$$\mathcal{E}(t) = \frac{1}{2}\epsilon_0 E^2(t) \cdot \pi r^2 l =$$

$$B(t) = \frac{r}{2}\frac{1}{c^2}\frac{dE(t)}{dt}$$

$$S = \frac{1}{\mu_0}\frac{1}{c^2}\frac{r}{2} E(t)\frac{dE}{dt} = \epsilon_0 \frac{r}{2} E(t) \frac{dE}{dt}$$

$$\frac{d\mathcal{E}}{dt} = 2\pi r \cdot l \cdot S$$

$$= \pi r^2 l \, \epsilon_0 \, E(t) \frac{dE(t)}{dt}$$

$$= \frac{d}{dt}\left(\frac{1}{2}\epsilon_0 E^2 \cdot \text{Vol} \right) \quad \checkmark$$]

3.) $$m(\ddot{x} + \omega_0^2 x - \eta \dddot{x}) = 0$$

a.) Method 1:

the energy of the oscillator is $E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$

$$\frac{dE}{dt} = m(\ddot{x} + \omega_0^2 x)\dot{x} = + m\eta \dot{x} \dddot{x}$$

If $x(t) = A(t) \cos \omega t$ $\qquad E(t) = \frac{1}{2} m \omega_0^2 A^2(t)$

$$\frac{d}{dt} E = \frac{d}{dt}\left(\frac{1}{2} A^2(t)\right) m\omega_0^2 = m\eta \left[-\omega_0^4 \sin^2 \omega_0 t\right] A^2(t)$$

average over a cycle.

$$\frac{1}{2} m\omega_0^2 \cdot \frac{d}{dt} A^2(t) = - m\omega_0^4 \cdot \frac{1}{2} \eta A^2(t)$$

$$\frac{d}{dt}\left(A^2(t)\right) = - \eta \omega_0^2 A^2(t)$$

$$A^2(t) = A_0^2 e^{-\eta \omega_0^2 t}$$

$$A(t) = A_0 e^{-\frac{\eta \omega_0^2}{2} t}$$

Method 2:

$$\ddot{x} + \omega_0^2 x - \eta \dddot{x} = 0$$

Form transform $\left[-\omega^2 + \omega_0^2 - \eta(-i\omega)^3\right] \tilde{x}(\omega) = 0$

$$\omega^2 = \omega_0^2 - i\eta\,\omega^3$$

so $$\omega^2 \approx \omega_0^2 - i\eta\omega_0^3$$

$$\omega \cong \pm\omega_0 - i\eta\frac{\omega_0^2}{2}$$

so solutions are

$$x(t) \cong A\, e^{-\frac{\eta\omega_0^2}{2}t} \cos\omega_0 t$$

b.) The retarded Green's function obeys:

$$\left(\frac{d^2}{dt^2} + \omega_0^2 - \eta\frac{d^3}{dt^3}\right) G_R(t) = \delta(t)$$

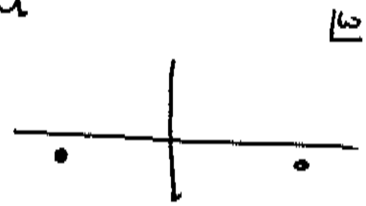
Fourier transform

$$\left(-\omega^2 + \omega_0^2 - i\eta\omega^3\right) \tilde{G}_R(\omega) = 1$$

$$\tilde{G}_R(\omega) = \frac{1}{-\omega^2 + \omega_0^2 - i\eta\omega^3}$$

The denominator has three poles. Two are at

$$\omega \cong \pm\omega_0 - i\tfrac{1}{2}\eta\omega_0^2$$



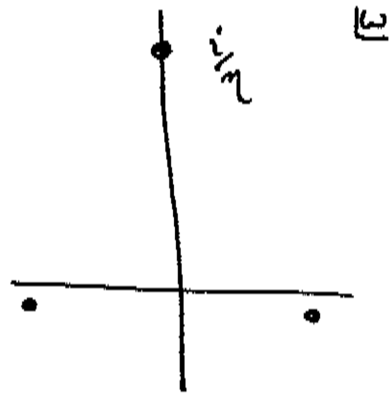
leading to a damped behavior. Where is the third?

If ω is large

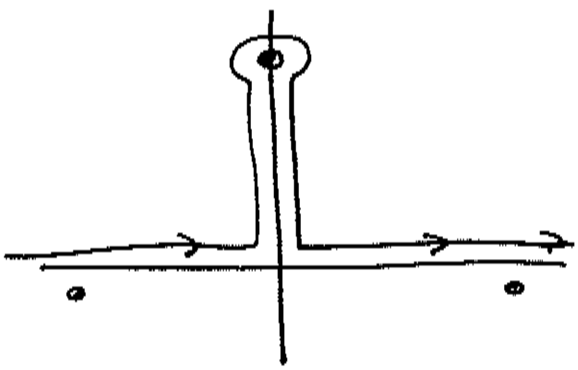
$$-\omega^2 - i\eta\omega^3 \cong 0$$

$$-i\eta\omega \cong 1$$

$$\omega \cong \frac{+i}{\eta}$$



The retarded Green's function is

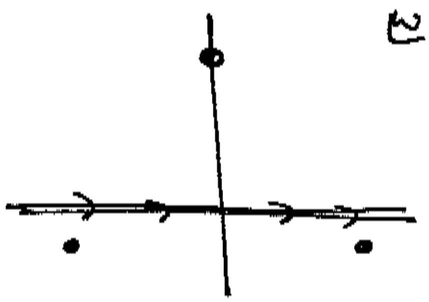


w. behavior

$$G_R(t) \sim e^{+t/\eta}$$

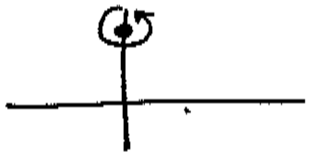
as $t \to \infty$

Abraham and Lorentz advocated using the Green's function

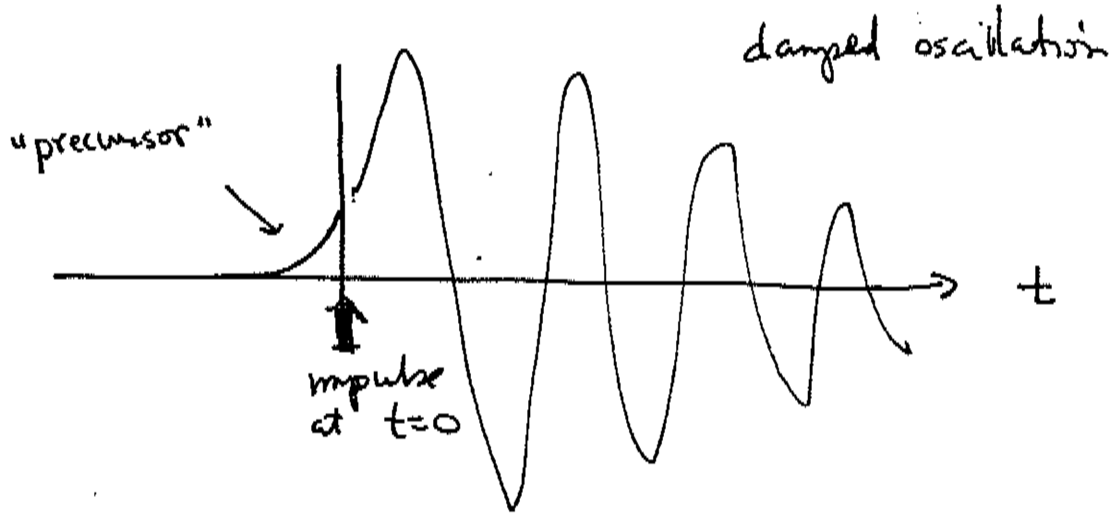


this function has odd acausal behavior

for $t<0$

$G_{AL}(t) =$  $\sim e^{-|t|/\eta}$

i.e.



A+L hoped ~~that~~ η is sufficiently small that this solution would not lead to acausal wierdness.

Today, this behavior is excluded.

4.) a.) The stress tensor σ^{ij} must be a tensor, that is, it must naturally have 2 vector indices, and it must be symmetric in these indices.

So

$$\sigma^{ij} \sim A\,\delta^{ij} + B\left(\frac{\partial}{\partial x^i}v^j + \frac{\partial}{\partial x^j}v^i\right)$$

are possible structures, $\sigma^{ij} \sim v^k$ or $\epsilon^{ijk}v^k$ is not. $A = -p = -$ (pressure)

Actually $\sigma^{ij} \sim C\,v^i v^j$ is a possible structure. Given that A, B, C arise from the atomic properties of the fluid

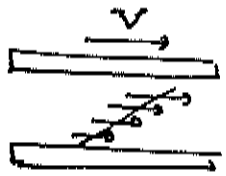
$B \sim A \cdot T$ where T is an atomic time

$C \sim A \cdot (R/T)^{-2}$ R/T is an atomic velocity

Under typical non relativistic conditions, B is small but non negligible, C is unimportant

[Sorry, this was more sophisticated than I intended.]

b.) The top plate moves with velocity $\vec{V}_0$. The fluid moves with



$$\vec{V}(z) = \vec{V}_0 \cdot \left(\frac{z}{h}\right) \qquad \vec{V}_0 \parallel \hat{x}$$

the $\hat{x}$ force on the top plate is

$$(\text{Force} / \text{Area}) = \sigma^{xi} (-\hat{z})^i = -\sigma^{xz} = -\mu \frac{V_0}{h}$$

so

$$M \ddot{x} = -\mu \frac{A}{h} \dot{x}$$

c.) Add a force turned on at $t=0$

$$M \ddot{x} + \mu \frac{A}{h} \dot{x} = F$$

the solution is:

$$\dot{x}(t) = \frac{Fh}{\mu A} \left[1 - e^{-\frac{\mu A}{Mh} t} \right]$$

with

$$\ddot{x}(0) = \frac{F}{M} \qquad \dot{x} \to \frac{Fh}{\mu A} \text{ as } t \to \infty$$