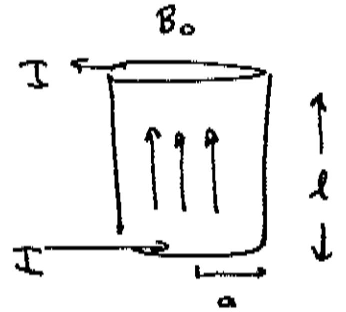
Physics 121 – Final Exam

Solutions

1.) The magnetic field in the solenoid is



$$B_0 = \mu_0 I n$$

a.) Energy stored in the field $= \int d^3x \; \frac{1}{2\mu_0} B_0^2$

$$= \pi r^2 l \; \frac{1}{2\mu_0} B_0^2$$

If r is increased, keeping I fixed, this relation continues to hold. Then

$$\frac{\partial \mathcal{E}}{\partial r} = 2\pi r l \cdot \frac{1}{2\mu_0} B_0^2$$

$$= (\text{Area of Surface}) \cdot \frac{B_0^2}{2\mu_0}$$

since this must be $\frac{d\text{Work}}{dr}$ = Area · pressure. $\therefore$ $P = \frac{B_0^2}{2\mu_0}$

b.) Inside the solenoid, the electromagnetic stress tensor is given by:

$$T^{ij} = \frac{1}{\mu_0} B^i B^j - \frac{1}{2} \delta^{ij} \left(\frac{1}{2\mu_0} B_0^2\right)$$

if $\vec{B}_0 \parallel \hat{z}$

$$= \begin{pmatrix} -\frac{1}{2\mu_0}B_0^2 & & \\ & -\frac{1}{2\mu_0}B_0^2 & \\ & & +\frac{1}{2\mu_0}B_0^2 \end{pmatrix}$$

the stress tensor is $\sigma^{ij} = -T^{ij}$, so

$$\sigma^{xx} = \sigma^{yy} = +\frac{1}{2\mu_0} B_0^2$$

at a place on the wall with $\hat{n} \parallel \hat{x}$, the pressure is

$$\sigma^{xx} = \frac{1}{2\mu_0} B_0^2$$

by symmetry, the pressure is the same all around.

c) The magnetic force on a wire is

$$\langle B \rangle \, I \cdot dl \qquad \text{outward.}$$

where $\langle B \rangle$ is a suitable average of $B = B_0$ inside and $B = 0$ outside. We can just write $\langle B \rangle = \frac{1}{2} B_0$, or (to be more sophisticated) note that the B of a current sheet is

$$\tfrac{1}{2}\mu_0 \mathcal{K} \uparrow \quad \times \quad \tfrac{1}{2}\mu_0 \mathcal{K} \downarrow$$

so $B = \frac{1}{2} B_0 \uparrow +$ (self-field of current sheet.)

We'll use the external field $\frac{1}{2} B_0$ to compute the force.

Anyway

$$\text{Force/area} = \frac{1}{2} B_0 \cdot I \cdot n = \frac{1}{2\mu_0} B_0^2$$

d) $$P = \frac{1}{2\mu_0} B_0^2 = \frac{1}{2} \frac{1}{4\pi \times 10^{-7}\ N/A^2} \cdot \left(10 \ \frac{N}{Am}\right)^2$$

$$= 4.0 \times 10^{7} \ N/m^2$$

2.) The retarded Green's function obeys the equation

$$m \frac{d^2}{dt^2} G_R(t) + k\, G_R(t-\tau) = \delta(t)$$

with $G_R(t) = 0$ for $t<0$

a.) Introduce $G_R(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \tilde{G}_R(\omega)$

$$m \frac{d^2}{dt^2} G_R(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, (-m\omega^2)\, \hat{G}_R(\omega)$$

$$k\, G_R(t-\tau) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, k\, e^{i\omega\tau}\, \hat{G}_R(\omega)$$

$$\delta(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}$$

so

$$(-m\omega^2 + k\, e^{i\omega\tau})\, \tilde{G}_R(\omega) = 1$$

$$\tilde{G}_R(\omega) = \frac{-1}{(m\omega^2 - k\, e^{i\omega\tau})}$$

$$G_R(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{-1}{(m\omega^2 - k\, e^{i\omega\tau})}$$

b) Where are the poles?

$$m\omega^2 = k e^{i\omega\tau} \Rightarrow \omega \cong \pm\sqrt{\frac{k}{m}} = \pm\omega_0$$

we expand $\quad \omega = \sqrt{\frac{k}{m}}\left(1 + \frac{i}{2}\omega\tau + \cdots\right)$

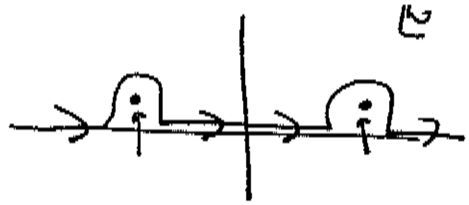
so $$\omega = \sqrt{\frac{k}{m}}\left(1 + \frac{i}{2}\sqrt{\frac{k}{m}}\,\tau + \cdots\right) = \omega_0 + i\frac{\omega_0^2}{2}\tau + \cdots$$

or

$$\omega = -\sqrt{\frac{k}{m}}\left(1 - \frac{i}{2}\sqrt{\frac{k}{m}}\,\tau + \cdots\right) = -\omega_0 + i\frac{\omega_0^2}{2}\tau + \cdots$$

Notice that both poles are in the upper half plane. So, the system is unstable.

c.) $$G_R(t) = \int\frac{d\omega}{2\pi}\, e^{-i\omega t}\,\frac{-1}{m\omega^2 - k e^{i\omega\tau}}$$



$$= 0 \qquad t<0$$

$$= \frac{1}{2\pi}\frac{(-1/m)\,(-2\pi i)}{2\omega_0 - ik\tau}\, e^{-i\omega_0 t}\, e^{+\frac{\omega_0^2}{2}\tau t}$$

$$+ \frac{1}{2\pi}\frac{(-1/m)\,(-2\pi i)}{-2\omega_0 - ik\tau}\, e^{+i\omega_0 t}\, e^{\frac{\omega_0^2\tau t}{2}}$$

$$\sim \frac{1}{m\omega_0}\sin\omega_0 t \cdot e^{+\left(\frac{\omega_0^2\tau}{2}\right)t}$$

So

d.) The system runs away from equilibrium in a time

$$t \sim \left(\frac{2}{\omega_0^2 \tau} \right)$$

3.) a.) If we look for a wave solution for $\vec{E}$ in the superconductor

$$\vec{E} = \mathrm{Re}\; E_0\,\hat{x}\; e^{-i\omega t + ikz}$$

we find

$$\left(-\frac{\omega^2}{c^2} + k^2\right) E_0 = -\frac{1}{\lambda^2} E_0$$

$$k^2 = -\left(\frac{1}{\lambda^2} - \omega^2/c^2\right)$$

$$k = +i\left[\frac{1}{\lambda^2} - \omega^2/c^2\right]^{1/2}$$

Choose $+i$ so that the wave dies out as $z \to \infty$

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} \quad \Rightarrow \quad B_0 = \mathrm{Re}\;\vec{B}_0\, e^{-i\omega t + ikz}$$

with

$$\vec{B}_0 = \frac{\vec{k}\times\hat{x}}{\omega} = i\left(\frac{1}{\lambda^2\omega^2} - \frac{1}{c^2}\right)^{1/2}\hat{y}$$

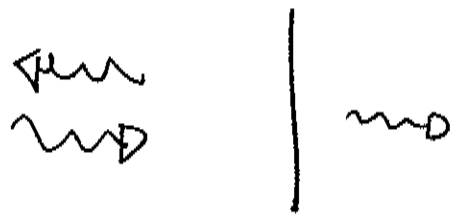
again:

$$\vec{E} = \mathrm{Re}\; E_0\,\hat{x}\; e^{-i\omega t - \varkappa z}$$

$$\vec{B} = \mathrm{Re}\; E_0\frac{i\varkappa}{\omega}\,\hat{y}\; e^{-i\omega t - \varkappa z}$$

with $\varkappa = \left[\frac{1}{\lambda^2} - \omega^2/c^2\right]^{1/2}$

b.) Now solve the problem of scatt at normal incidence:



$z < 0$

$$\vec{E} = \mathrm{Re}\; E_0 \left(\hat{x}\, e^{-i\omega t + ikz} + R\, \hat{x}\, e^{-i\omega t - ikz} \right)$$

$$\vec{B} = \mathrm{Re}\; \frac{E_0}{c} \left(\hat{y}\, e^{-i\omega t + ikz} - R\, \hat{y}\, e^{-i\omega t - ikz} \right)$$

$z > 0$

$$\vec{E} = \mathrm{Re}\; E_0\, T\, \hat{x}\, e^{-i\omega t - \kappa z}$$

$$\vec{B} = \mathrm{Re}\; E_0 T \frac{ik}{\omega}\, \hat{y}\, e^{-i\omega t - \kappa z}$$

Impose the conditions ($\varepsilon = \varepsilon_0$ $\mu = \mu_0$ inside)

that $E + B$ are continuous at $z = 0$

$$E: \quad 1 + R = T$$

$$B: \quad \frac{1}{c}(1 - R) = \frac{ik}{\omega} T$$

so

$$1 + R = T \qquad 1 - R = \frac{i\kappa c}{\omega} T$$

$$T = \frac{2}{1 + \frac{i\kappa c}{\omega}} \qquad R = \left(\frac{1 - \frac{i\kappa c}{\omega}}{1 + \frac{i\kappa c}{\omega}} \right)$$

with, again, $\quad \kappa = \left[\frac{1}{a^2} - \omega^2/c^2 \right]^{1/2}$

$$|R|^2 = 1$$

$R = e^{i\Phi}$ where $\quad \tan \Phi/2 = -\frac{\kappa c}{\omega}$

as $\omega \to 0 \quad \tan \Phi/2 \cong -\frac{1}{a}\frac{c}{\omega}$

$$\cot \Phi/2 \cong -ka$$

which $\to 0$ as $\omega \to 0$

so

$$R \approx (-1) \cdot e^{-2ika}$$

as $\omega \to 0$

c.) In the superconductor $\langle \vec{S} \rangle = \langle \vec{E} \times \vec{B} \rangle = 0$
so all of the energy that comes in in the initial wave
must go out in the reflected wave. Thus,

$$|R| = 1$$

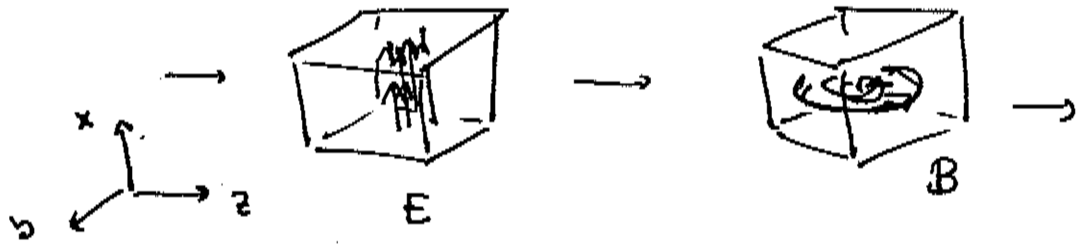
d.) For $\omega \to 0$ $R = -1$ as expected
for a perfect conductor. Near this limit, the
wave goes an extra distance λ into the superconductor.
This gives an extra phase factor

$$\left(e^{-ik\lambda} \right)^2$$

$\uparrow$
in and out

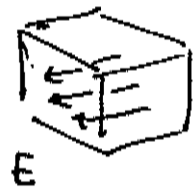
4.) a.) The lowest-frequency mode in a cavity of size $a \times a \times a$ has

$$\vec{E}_{(x)} = \hat{x} E_0 \sin\frac{\pi y}{a} \sin\frac{\pi z}{a} \cos\omega t$$



or

$$\vec{E}_{(y)} = \hat{y} E_0 \sin\frac{\pi x}{a} \sin\frac{\pi z}{a} \cos\omega t$$



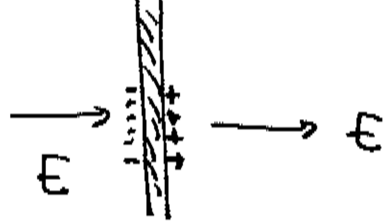
or

$$\vec{E}_{(z)} = \hat{z} E_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \cos\omega t$$

all with

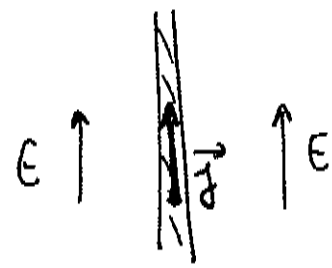
$$\omega^2/c^2 = 2\left(\frac{\pi}{a}\right)^2$$

b) The component of $\vec{E}$ normal to the sheet will cause a small charge separation



requiring, if the sheet is thin, only a tiny current.

However, the component of $\vec{E}$ parallel to the sheet will drive a current along the sheet



so $\vec{j} = \sigma \vec{E}_{\parallel}$

The work done on the sheet per unit volume per sec $= \vec{j} \cdot \vec{E}$

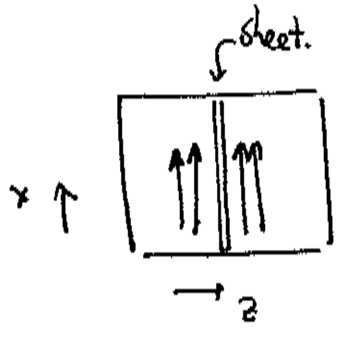
$$= \sigma E_{\parallel}^2$$

so

$$\text{Energy dissipated/sec} = \int d(\text{Area}) \cdot d \cdot \sigma E_{\parallel}^2$$

c.) Looking into the cube from the face $\perp \hat{y}$

for $E(x)$

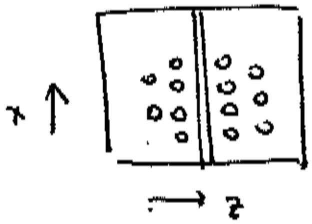


on the sheet $\quad E_{\parallel} = E_0 \sin \frac{\pi y}{a} \cdot 1 \cdot \cos \omega t$

$$\left\langle \int d^2a \; E_{\parallel}^2 \right\rangle = \frac{1}{4} E_0^2 \cdot a^2$$

so $$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = -\frac{1}{4} a^2 d \sigma E_0^2$$

for $E_{(y)}$



$$E_{\parallel} = E_0 \sin\frac{\pi x}{a} \cdot 1 \cdot \cos \omega t$$

so $\frac{d\mathcal{E}}{dt}$ is the same

for $E_{(z)}$ $\vec{E} \parallel \hat{z}$ and $\perp$ to the sheet.

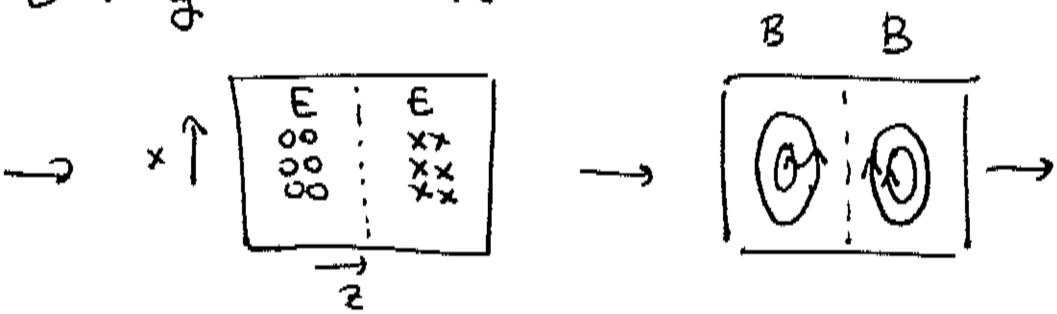
so:

$E_{(x)}$, $E_{(y)}$ $$\frac{d\mathcal{E}}{dt} = -\frac{1}{4} a^2 d \sigma E_0^2$$

$E_{(z)}$ $$\frac{d\mathcal{E}}{dt} = 0$$

d.) The mode with $\omega^2/c^2 = 5\pi^2/a^2 = \left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2$

with $\vec{E} \parallel \hat{y}$ looks like:



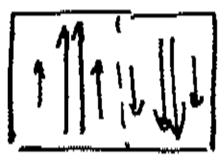
actually, there is another mode with $\vec{E} \parallel \hat{y}$ that looks like this:

$\mathcal{E}$ $\qquad\qquad$ B

in all, there are $\underline{6}$ modes with $\omega^2/c^2 = 5\,\pi^2/a^2$

Of these, the two with $\vec{E} \parallel \hat{z}$ have $\underline{\text{zero}}$ power dissipation. Also, of the two modes above, the first has $\vec{E} = 0$ on the sheet $\Rightarrow$ $\dfrac{d\mathcal{E}}{dt} = 0$

There is a similar mode with $\vec{E} \parallel \hat{x}$:



so, of the $\underline{6}$ modes, four have $\dfrac{d\mathcal{E}}{dt} = 0$

$\Rightarrow$ this approximation.

5.) a.) Cylindrically symmetric waveforms obeying Neumann boundary conditions on the wall of a pipe have the form

$$p(t,\vec{x}) = \mathrm{Re}\left[p_0 \; J_0(k_\perp r)\, e^{-i\omega t}\, e^{ikz} \right]$$

where $\quad k_\perp a = $ max. or min of $J_0(z)$

$$\frac{\omega^2}{c^2} = k^2 + k_\perp^2$$

the possible values of $k_\perp$ are $\quad \frac{0}{a}, \; \frac{3.83}{a}, \; \ldots$

for the second choice

$$\omega_c^2 = c^2 k_\perp^2 = \left[(300\,\mathrm{m/sec})\cdot\frac{3.83}{5\mathrm{m}}\right]^2 = \left[230/\mathrm{sec}\right]^2$$

this is the cutoff frequency for that mode, which should be compared to the frequency of the drum beat

$$\omega = \frac{2\pi}{T} = 2\pi\cdot 5/\mathrm{sec} = 3[redacted]./\mathrm{sec}.$$

since $\omega_c > \omega$, the second mode does not propagate down the pipe, so the waveform at a large distance down the pipe is just

$$p(t,\vec{x}) = \mathrm{Re}\left[p_0 \; e^{-i\omega t}\, e^{ikz}\right]$$

with $\omega = ck$. The signals propagate at the speed of sound, $v = c = 300\ \mathrm{m/sec}$

b.) If the drumhead is hit off-center, it is possible to excited the mode:

$$p(t,\vec{x}) = \mathrm{Re}\left[P_1 \, J_1(k_\perp r)\cos\phi \, e^{-i\omega t + ikz} \right]$$

To satisfy the b.c.'s. $\quad k_\perp a$ = min or max of $J_1(z)$

$$= 1.84,\ 5.33,\ \ldots$$

for the lowest value, the cutoff frequency is

$$\omega_c = ck_\perp = 300\ \mathrm{m/sec} \cdot \frac{1.84}{5\ \mathrm{m}}$$

$$= 110.\ /\mathrm{sec}$$

to be compared to

$$\omega = \frac{2\pi}{T} = 2\pi \cdot \underline{20/\mathrm{sec}} = 126/\mathrm{sec}.$$

so the lowest such mode propagates in the pipe. The complete waveform is

$$p(t,z) = \mathrm{Re}\Big[\; p_0 \; e^{2i\omega t + ik_0 z}$$

$$+ \; p_1 \cos\phi J_1\left(\frac{1.84}{a} r\right) e^{-i\omega t} e^{ik_1 z} \Big]$$

where $\omega = 126\ /\mathrm{sec}$ $\quad T = 1/20\ \mathrm{sec}$

$2\omega = 251\ /\mathrm{sec}$ $\quad T = 1/40\ \mathrm{sec}$

$k_0 = 2\omega/c$ $\qquad k_1 = \left[\omega^2/c^2 - k_\perp^2\right]^{1/2}$

the 0 wave propagates at $c = 300\ \mathrm{m/sec}$

the 1 wave propagates at the group velocity

$$\frac{\partial \omega}{\partial k} = \frac{k}{\omega} = c\left[1 - \frac{c k_\perp^2}{\omega^2}\right]^{1/2}$$

$$= c\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{1/2}$$

$$= 146\ \mathrm{m/sec}$$

c.) to estimate the relative amplitudes of the waves, overlap the delta-function impulse at $r = a/2$ with the waveform.

$$\int d^2x \; \frac{1}{a/2} \delta(r - a/2) \, \delta(\phi)$$

$$= \int dr \, r \, d\phi \; \frac{1}{a/2} \delta(r - a/2) \, \delta(\phi) = 1$$

so this is the correct δ-function, Then

$$P_0 \sim \frac{\int d^2x \; \frac{1}{a/2} \delta(r - a/2) \, \delta(\phi) \cdot 1}{\int d^2x \quad \cdot (1)^2} = \frac{1}{\pi a^2}$$

$$P_1 \sim \frac{\int d^2x \; \frac{1}{a/2} \delta(r - a/2) \, \delta(\phi) \cdot J_1(k_\perp r) \cos\phi}{\int d^2x \quad \left[J_1(k_\perp r) \cos\phi \right]^2}$$

$$= \frac{J_1\left(k_\perp \frac{a}{2}\right)}{\int_0^a dr \, r \; J_1^2(k_\perp a) \cdot \pi} = \frac{1}{\pi a^2} \frac{J_1\left(\frac{1.84}{2}\right)}{\int_0^b dz \, z J_1^2(z)}$$

again:

$$\frac{P_1}{P_0} = \frac{J_1(y_{11}/2)}{\int_0^{y_{11}} dz\; z\; J_1^2(z)}$$

with $y_{11} = 1.84$.

6.) a,b) It is easiest to start with (b)

Let
$$\mathcal{E} = \tfrac{1}{2}\rho\left(\frac{\partial n^i}{\partial t}\right)^2 + \tfrac{1}{2}\kappa\,(\nabla^j n^i)^2$$
$$+\tfrac{1}{2}\mu\,(1-|\vec{n}|^2)^2$$

be the energy density

$$\frac{\partial \mathcal{E}}{\partial t} = \rho\,\frac{\partial n^i}{\partial t}\,\frac{\partial^2 n^i}{\partial t^2} + \kappa\,(\nabla^j n^i)\,\nabla^j\!\left(\frac{\partial n^i}{\partial t}\right)$$
$$+ (-2\mu\, n^i)(1-|\vec{n}|^2)\,\frac{\partial n^i}{\partial t}$$

$$= \frac{\partial n^i}{\partial t}\Big\{\rho\,\frac{\partial^2 n^i}{\partial t^2} - \kappa\,\nabla^2 n^i$$
$$- 2\mu\, n^i(1-|\vec{n}|^2)\Big\}$$
$$+ \kappa\,\nabla^j\left((\nabla^j n^i)\,\frac{\partial n^i}{\partial t}\right)$$

to get the right equation for local conservation

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla^j J_{\mathcal{E}}^j = 0$$

the quantity in braces {} must vanish. So this should be the equation of motion (3). And it is, if

$$K = c^2 \rho$$

$$\mu = \tfrac{1}{2} \rho c^2 / \ell^2$$

then

$$\vec{j}_\mathcal{E} = - \rho c^2 \frac{\partial n^i}{\partial t} \vec{\nabla} n^i$$

and

$$\frac{d}{dt} \int d^3x \ \mathcal{E} = 0$$

c.) $\mathcal{E}$ is a sum of squares, so to minimize the energy each term should vanish individually.

$$\frac{\partial n^i}{\partial t} = 0 \qquad \vec{\nabla} n^i = 0 \qquad 1 - |\vec{n}|^2 = 0$$

so $\quad |\vec{n}| = 1 \quad$ & constant everywhere.

d.) For $\vec{n}(x) = (0, \delta n, 1)$, the equation for $n^y(t, \vec{x})$ is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta n = \frac{1}{\ell^2}\,\delta n\,\left(1 - [1 + (\delta n)^2]\right)$$

$$= \Theta((\delta n)^3)$$

so for small δn

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta n = 0$$

and $\omega(k) = ck$

e.) For $\vec{n}(x) = (0, 0, 1 + \delta n)$

the equation for n^3 is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)(1 + \delta n) = \frac{1}{\ell^2}(1 + \delta n) \cdot \left(1 - (1 + \delta n)^2\right)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta n = \frac{1}{\ell^2}(1 + \delta n) \cdot \left[-2\delta n + (\delta n)^2\right]$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta n = -\frac{2}{\ell^2}\,\delta n + \mathcal{O}((\delta n)^3)$$

a wave solution $\quad \delta n = \mathrm{Re}\left[\alpha_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \right]$

satisfies

$$\left[-\frac{\omega^2}{c^2} + k^2 = -\frac{2}{\ell^2} \right] \alpha_0$$

$$\omega(k) = c\left[2/\ell^2 + |\vec{k}|^2 \right]^{1/2}$$

f.) For $\vec{n}(x) = (0\ 0\ \delta n)$

the equation for n^3 is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \delta n = \frac{1}{\ell^2}\,\delta n + \mathcal{O}((\delta n)^2)$$

Look for a solution $\quad \delta n = \mathrm{Re}\left[\alpha\, e^{-i\omega t} \right] \quad (k=0)$

$$\left[-\frac{\omega^2}{c^2} = +\frac{1}{\ell^2} \right] \delta n$$

so there is an <u>unstable</u> solution $\quad \delta n \sim e^{+\frac{c}{\ell}t}$

This makes sense, because $\vec{n} = 0$ is a <u>maximum</u> of the energy.