

Physics 121 – Final Exam

(Wednesday, June 12)

There are six problems, four worth 15 points each and two somewhat longer ones worth 20 points each. The total point score on the exam is 100 points.

- (15 points) Consider a cylindrical solenoid of radius a carrying inside it a uniform magnetic field B_0 . The solenoid is built of a large number n of windings per m, each of which carries a current I . Compute the pressure on the wall in three ways:
 - Compute the stored magnetic energy as a function of the radius, assuming that the current stays constant as the radius is changed.
 - Compute the stress tensor just inside the wall.
 - Compute the force of the magnetic field acting on the current in the wall. You will need to take into account that the magnetic field falls to zero across the wires.
 - Finally, get a numerical value for the pressure, in N/m^2 , for a solenoid of radius 10 cm carrying a 10 T field.
- (15 points) A servomechanism is supposed to control an object of mass m to stay near the position $x = 0$. Unfortunately, there is some delay in the circuit that controls the servo, so that the mass obeys the equation of motion

$$m\ddot{x}(t) = -kx(t - \tau) , \quad (1)$$

where τ is a fixed delay.

- Using the Fourier transformation, construct the Green's function for this equation as an integral over the Fourier variable ω .
 - To first order in τ , solve for the positions of the poles of the Fourier transform.
 - Continuing to use this approximation, find an expression for the Green's function as a function of t .
 - Does the servomechanism work as intended?
- (15 points) Consider the reflection of an electromagnetic wave from a superconductor. For this exam, you can treat a superconductor as a plasma with no energy dissipation. The electric and magnetic fields obey the equations:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} = 0 , \quad \vec{\nabla} \cdot \vec{B} = 0 , \quad \vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \\ \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{E} = -\frac{1}{\lambda^2} \vec{E} \end{aligned} \quad (2)$$

where λ is the 'penetration depth'.

- (a) Find a wave solution in the superconductor for the case $\omega < c/\lambda$. Write out $\vec{E}(t, \vec{x})$, $\vec{B}(t, \vec{x})$ as real-valued fields.
- (b) Solve the problem of reflection of an electromagnetic wave in vacuum with $\omega < c/\lambda$ from the surface of a superconductor at normal incidence. Compute the (complex) reflection coefficient R . Show that $|R|^2 = 1$, and compute the phase of R .
- (c) Why is $|R|^2 = 1$?
- (d) What is the significance of the phase of R ?
4. (20 points) Consider a cubic cavity with sides of length a , made of perfectly conducting material. I insert inside the cavity a thin sheet of material with $\epsilon \approx \epsilon_0$, $\mu \approx \mu_0$ which is a poor conductor. The sheet is of size $a \times a$ and of thickness $d \ll a$; the conductivity is σ . I place this sheet perpendicular to the \hat{z} direction so that it just bisects the cavity.
- (a) Ignoring the sheet for a moment, the cavity has three modes of electromagnetic oscillation with $\omega^2 = 2c^2\pi^2/a^2$. Write $\vec{E}(t, \vec{x})$ for each mode.
- (b) I claim that, since the sheet is thin, an \vec{E} field will drive currents in the plane of the sheet only, i.e., $\vec{j} = \sigma\vec{E}_{\parallel}$. Why? For a given value of \vec{E}_{\parallel} , how much energy is dissipated per unit area per second?
- (c) For each of the modes in part (a), compute the total power dissipation averaged over a cycle. You can take the \vec{E} field to be just that computed in part (a), ignoring the sheet.
- (d) The cavity has six modes of electromagnetic oscillation with $\omega^2 = 5c^2\pi^2/a^2$ (e.g., TM_{210}). Sketch them. Which have zero power dissipation in the approximation used in this problem?
5. (15 points) Imagine a deep mine, with a cafeteria on the surface. A drummer signals when it is time for lunch by beating on a drumhead fastened across a large cylindrical pipe that goes down into the mine. Describe waves in the pipe by taking the pressure $p(x)$ (a scalar quantity) to obey the wave equation, with Neumann boundary conditions ($\partial p/\partial r = 0$) on the wall of the pipe. Take the speed of sound to be 300 m/sec and the radius of the pipe (and the drumhead) to be $a = 5$ m.
- (a) The drummer hammers on the center of the drum, hitting it 5 times/sec. This causes waves to propagate down the pipe. One possible waveform is a wave that is uniform across the pipe; other possible waveforms have the behavior $J_0(kr)$ across the cross section for some values of $k > 0$. Find the lowest nonzero value of k other than $k = 0$. Taking note of this value, find the waveform very far down the pipe. At what speed do the signals propagate down the pipe? Give a numerical answer.
- (b) A very energetic drummer hammers the drum at 40 times/sec, in such a way that he hits the drumhead alternately at the positions $(a/2, 0, 0)$ and $(-a/2, 0, 0)$.

Show that the signal now propagates down the pipe as two signals travelling at different speeds. Find the waveforms for the two signals. Find, numerically, the two speeds of propagation.

- (c) Find the relative amplitudes of the two signals. (In this part, you can express your answer in terms of a ratio of values of $J_m(z)$ for different values of m and z .)
6. (20 points) A *nematic liquid crystal* is a fluid containing long molecules, such that the properties of the fluid depend on the orientation of the molecules. Let $\hat{n}(t, \vec{x})$ be a vector representing the average orientation of the molecules. An appropriate equation of motion for \vec{n} is:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right) \vec{n} = \frac{1}{\ell^2} \vec{n} (1 - |\vec{n}|^2) \quad (3)$$

- (a) Show that the following expression is conserved by virtue of this equation for an appropriate choice of ρ , κ , μ ,

$$E = \int d^3x \left\{ \frac{1}{2} \rho \left(\frac{\partial n^i}{\partial t} \right)^2 + \frac{1}{2} \kappa (\nabla^j n^i)^2 + \frac{1}{2} \mu (1 - |\vec{n}|^2)^2 \right\} \quad (4)$$

and find the appropriate values of κ , μ in terms of the ρ , c , and ℓ .

- (b) Show that the energy is locally conserved, and find the energy current \vec{j}_E .
- (c) Show that the energy is minimized by a state with $|\vec{n}| = 1$ everywhere.
- (d) Consider a state with $\vec{n}(x) = (0, \delta n(t, \vec{x}), 1)$. Plug this into eq. (3) and linearize in δn . Find the wave solutions of the resulting linear equation. What is the dispersion relation $\omega(\vec{k})$?
- (e) Consider a state with $\vec{n}(x) = (0, 0, 1 + \delta n(t, \vec{x}))$. Plug this into eq. (3) and linearize in δn . Find the wave solutions of the resulting linear equation. What is the dispersion relation $\omega(\vec{k})$?
- (f) Plug the configuration $\vec{n}(x) = (0, 0, \delta n(t, \vec{x}))$ into eq. (3), and linearize the equation. Can you find a wave solution? For clarity, try to find a solution with $\vec{k} = 0$. What happens?

Some useful quantities:

$$\begin{aligned}\epsilon_0 &= 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2 \\ \mu_0 &= 4\pi \times 10^{-7} \text{ N/A}^2 \\ e &= 1.60 \times 10^{-19} \text{ C} \\ c &= 3.00 \times 10^8 \text{ m/sec}\end{aligned}\tag{5}$$

Zeros of Bessel functions $J_m(z)$:

J_0	2.405	5.520	8.654
J_1	3.832	7.016	10.173
J_2	5.136	8.417	11.620

Zeros of $J'_m(z)$:

J_0	0.000	3.832	7.016
J_1	1.841	5.331	8.536
J_2	3.054	6.706	9.970