# The Wave Equation

We have now spent considerable effort study the wave solutions of Maxwell's equations. But, before we discuss applications of these solutions, I will like to go over this ground again, introducing some more useful mathematics.

Start again from Maxwell's equations for $\rho = \vec{J} = 0$.

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

(where I have put $\epsilon_0\mu_0 = \frac{1}{c^2}$). Combining the last two equations

$$\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{B} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{E})$$

$$= -\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) + \nabla^2\vec{E}$$

since $\vec{\nabla}\cdot\vec{E} = 0$, we see that $\vec{E}$ satisfies the equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E} = 0$$

By similar manipulations, it is easy to show that $\vec{B}$ satisfies the same equation. Notice that this equation does not group the vector index of $\vec{E}$. Each component of

$\vec{E}$ and $\vec{B}$ separately satisfies the scalar equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0$$

The vector character of $\vec{E}$ and $\vec{B}$ comes in when we impose the additional conditions

$$\vec{\nabla}\cdot\vec{E} = 0 \;, \quad \vec{\nabla}\cdot\vec{B} = 0$$

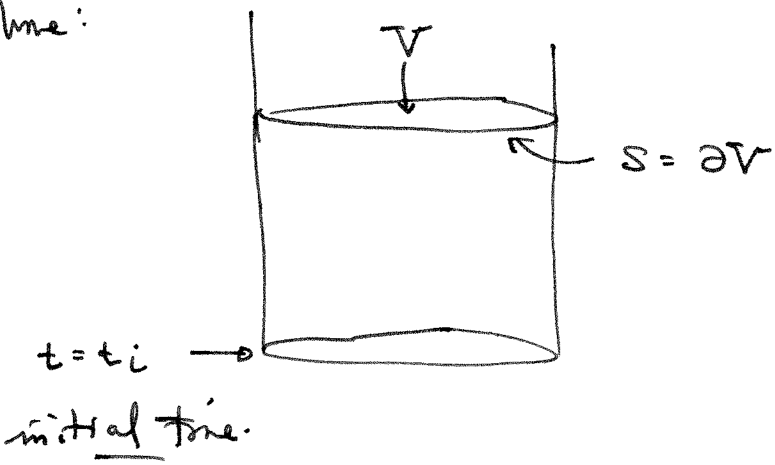
The equation above is called the wave equation (or, the "scalar wave equation"). The operator

$$\Box = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)$$

is called the wave operator or d'Alembertian.

Many aspects of the behavior of electromagnetic waves do not depend essentially on the vector character. To study these, we may use the scalar wave equation as a simpler model. For many problems in this course, we will first analyze the scalar wave equation and then discuss the complications which arise from the vector character of $\vec{E}$ and $\vec{B}$ and their various possible polarizations.

The wave equation is superficially similar to Poisson's equation, and again there is an existence and uniqueness theorem. For Poisson's equation, we saw that there existed a unique solution in a finite volume $V$ if the field $\phi(\vec{x})$ obeyed Dirichlet <u>or</u> Neumann boundary conditions at every point on the surface of $V$. Here the situation is more complicated. The wave equation also involves <u>time</u>, and we may think of the wave equation in a <u>spatial</u> volume $V$ as living in a space-time volume:




If the wave equation had time derivatives but no space derivatives, we could specify the initial conditions at $t_i$ and integrate these up to a unique solution for later times. An equation of second order in time requires <u>two</u> initial conditions: the values of $\phi$ and $\frac{d\phi}{dt}$ at $t = t_i$. For the wave equation, it turns out that a unique solution is picked out if we satisfy <u>both</u> the spatial and temporal boundary conditions:

$$(1) \quad \phi(x,t) = \phi_0(\vec{x}) \,, \quad \frac{\partial}{\partial t}\phi(x,t) = \dot{\phi}_0(\vec{x}) \quad \text{at } t = t_i$$

$$(2) \quad \phi(x,t) = 0 \quad \underline{\text{or}} \quad \hat{n}\cdot\vec{\nabla}\phi(x,t) = 0 \quad \text{for all } t$$

$$\text{on} \quad x \in \partial V .$$

To prove this assertion, we can analyze as for an ordinary differential equation in time: Discretize time, and show that $\phi(x,t)$, $\frac{\partial \phi}{\partial t}(x,t)$ can be found at $t = t_i + \epsilon$ from the values at $t = 0$. This solution requires solving a partial differential equation in space, but the solution to that equation is guaranteed if the boundary conditions (2) are satisfied at all times.

Here is another (very abstract) way to see that the wave equation has a unique solution in V: We can construct it! Under the condition (2), the Laplace equation is a Sturm Liouville problem in V, and we have a complete set of eigenfunctions

$$-\nabla^2 \varphi_i(\vec{x}) = \lambda_i \varphi_i(\vec{x})$$

with $\lambda_i \geq 0$. The initial conditions $\phi_0(\vec{x})$, $\dot{\phi}_0(\vec{x})$ can be expanded in this basis

$$\phi_0(\vec{x}) = \sum_i a_i \varphi_i(\vec{x})$$

$$\dot{\phi}_0(\vec{x}) = \sum_i b_i \varphi_i(\vec{x})$$

and $\phi(\vec{x},t)$ can be expanded at any time

$$\phi(\vec{x},t) = \sum_i c_i(t) \varphi_i(\vec{x})$$

Now the wave equation takes the form:

$$0 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi(\vec{x},t)$$

$$= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \sum_i c_i(t)\,\varphi_i(\vec{x})$$

$$= \sum_i \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \lambda_i\right] c_i(t)\ \varphi_i(\vec{x})$$

also, by orthogonality, for all $i$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \lambda_i\right) c_i(t) = 0$$

The solutions of this equation are

$$c_i(t) = C_i \cos(c\sqrt{\lambda_i}\,t) + D_i \sin(c\sqrt{\lambda_i}\,t)$$

we match the initial conditions if

$$C_i = a_i \qquad D_i \cdot c\sqrt{\lambda_i} = b_i$$

so:

$$\phi(x,t) = \sum_i \left[a_i \cos(c\sqrt{\lambda_i}\,t) + \frac{b_i}{c\sqrt{\lambda_i}} \sin(c\sqrt{\lambda_i}\,t)\right] \varphi_i(\vec{x})$$

which is an explicit solution as long as the series converges (which can be shown for smooth initial conditions).

This analysis is a little abstract, but we can make it more concrete by considering the special case:

$V =$ all of space, $\phi_0(x)$, $\dot{\phi}_0(x)$ are functions that $\to 0$ as $|\vec{x}| \to \infty$. Then we can represent $\phi_0$, $\dot{\phi}_0$ by their

Fourier transforms:

$$\phi_0(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\phi}_0(k)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\dot{\phi}_0(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\dot{\phi}}_0(k)\, e^{i\vec{k}\cdot\vec{x}}$$

and we can represent $\phi(x,t) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\phi}(k,t)\, e^{i\vec{k}\cdot\vec{x}}$

The wave equation now becomes

$$0 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \int \frac{d^3k}{(2\pi)^3}\, \tilde{\phi}(k,t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\phi}(k,t) + k^2\tilde{\phi}(k,t)\right] e^{i\vec{k}\cdot\vec{x}}$$

and, by orthogonality:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\phi}(k,t) + k^2\,\tilde{\phi}(k,t) = 0$$

the general solution of this equation is

$$\tilde{\phi}(k,t) = C(k)\cos(ckt) + D(k)\sin(ckt)$$

Matching to the Fourier coefficients of the initial condition, we have

$$\tilde{\phi}(k,t) = \tilde{\phi}_0(k)\cos(ckt) + \frac{\tilde{\dot{\phi}}_0(k)}{ck}\sin(ckt)$$

Thus, the wave equation in space, for any localized initial condition, has the general solution

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3} \left\{ \tilde{\phi}_0(k) \cos\omega t + \frac{\dot{\tilde{\phi}}(k)}{\omega} \sin\omega t \right\} e^{i\vec{k}\cdot\vec{x}}$$

with $\omega = c|\vec{k}|$. Notice that this is a linear combination of solutions of the form

$$e^{i\vec{k}\cdot\vec{x}}\, e^{\pm i\omega t} \qquad \omega = c|\vec{k}|$$

The same conclusion holds for Maxwell's equations; thus, the special wave solutions we have been studying are not so special, since any solution can be built by superposing them.

These general conclusions apply to any number of space dimensions. However, in <u>one</u> space dimension, the wave equation is especially simple. Note that, in 1-dimension

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)$$

These factors commute. So the most general solution of

$$\Box\phi(z,t) = 0$$

is a linear combination of factors which satisfy either

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\phi = 0 \quad \text{or} \quad \left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\phi = 0$$

The first condition is solved by

$$\phi(z,t) = f_R(z-ct)$$

The second condition is solved by

$$\phi(z,t) = f_L(z+ct)$$

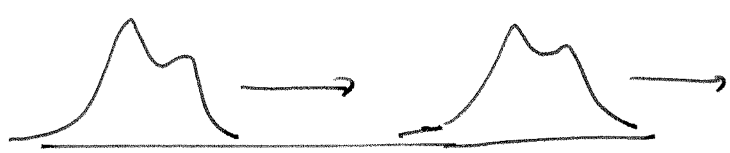
So in general

$$\phi(z,t) = f_R(z-ct) + f_L(z+ct)$$

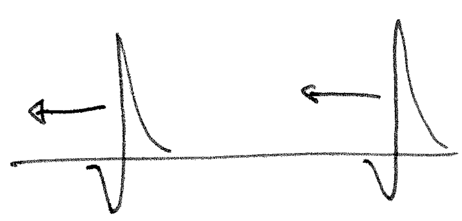
For example, if $\phi_0(z)$ is given and $\dot{\phi}_0(z) = 0$ at $t=0$

$$\phi(z,t) = \frac{1}{2}\left[\phi_0(z-ct) + \phi_0(z+ct)\right]$$

$f_R(z-ct)$ is a waveform that preserves its shape and moves to the right at speed $c$. (This is just the behavior we saw for the electromagnetic wavepacket bounded only in $z$.)

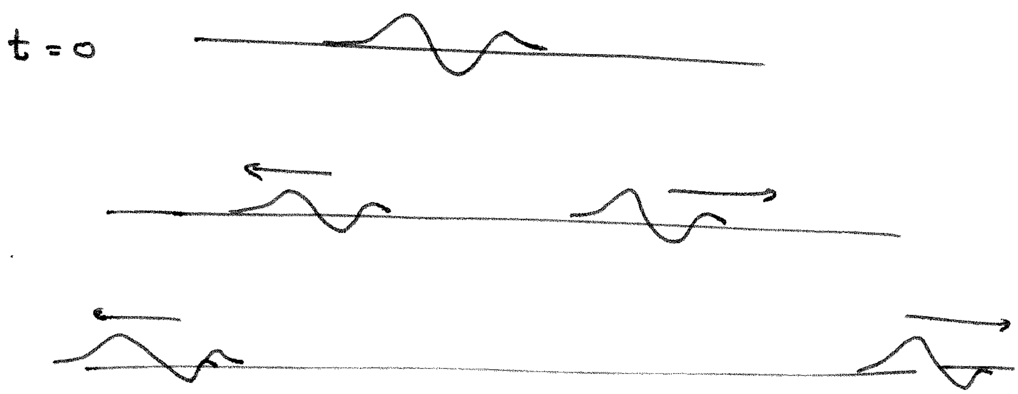


$f_L(z+ct)$ is a waveform that preserves its shape and moves to the left at speed $c$



The solution $\phi(z,t) = \frac{1}{2}\left[\phi_0(z-ct) + \phi_0(z+ct)\right]$

looks like:




All of the signals, all of the energy, and all of the information propagates exactly at the speed c. Though we will not find such simple solutions in higher dimensions, the conclusion that energy propagates at speed c will still be valid.

It will be useful to work out the energy and momentum associated with the scalar wave equation. A reasonable guess for $\mathcal{E}$ is that it is the sum of kinetic and strain energies of $\phi$:

$$\mathcal{E} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}c^2\left(\frac{\partial\phi}{\partial x_i}\right)^2$$

To check this, compute

$$\frac{\partial}{\partial t}\mathcal{E} = \frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2} + c^2\,\vec{\nabla}\phi\cdot\vec{\nabla}\frac{\partial\phi}{\partial t}$$

$$= c^2\,\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\vec{\nabla}\phi\right) + \frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial t}c^2\nabla^2\phi$$

so

$$\frac{\partial}{\partial t}\mathcal{E} = \vec{\nabla}\left[ c^2 \frac{\partial\phi}{\partial t}\vec{\nabla}\phi \right] + \frac{\partial\phi}{\partial t}\underbrace{\left(\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi\right)}_{=0}\cdot c^2$$

or

$$\frac{\partial}{\partial t}\mathcal{E} + \vec{\nabla}\cdot\vec{J}_{\mathcal{E}} = 0$$

with

$$\mathcal{E} = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}c^2\left(\vec{\nabla}\phi\right)^2$$

$$\vec{J}_{\mathcal{E}} = -\, c^2 \frac{\partial\phi}{\partial t}\vec{\nabla}\phi$$

For a simple wave solution

$$\phi(\vec{x},t) = \mathrm{Re}\,\phi_0\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \omega = c|\vec{k}|$$

$$\langle\mathcal{E}\rangle = |\phi_0|^2\cdot\frac{1}{2}\cdot\left[\frac{1}{2}\omega^2 + \frac{1}{2}c^2k^2\right]$$

$$= \frac{1}{2}\,\omega^2\,|\phi_0|^2$$

with equal contributions from kinetic and potential energy. To compute $\vec{J}_{\mathcal{E}}$, we must be more careful to get the sign right!

$$\phi(x,t) = \phi_0 \cos(\vec{k}\cdot\vec{x} - \omega t)$$

$$\frac{\partial\phi}{\partial t}(x,t) = \phi_0\,\omega \sin(\vec{k}\cdot\vec{x} - \omega t)$$

$$\vec{\nabla}\phi(x,t) = -\vec{k}\,\phi_0 \sin(\vec{k}\cdot\vec{x} - \omega t)$$

so
$$\vec{j}_\varepsilon = +\, c^2 \omega \vec{k}\, \phi_0^2 \sin^2(\vec{k}\cdot\vec{x} - \omega t)$$

$$\langle \vec{j}_\varepsilon \rangle = \left(\tfrac{1}{2}\omega^2 |\phi_0|^2\right)\cdot c \cdot \hat{k}$$

$$= \mathcal{E} c\, \hat{k}$$

For momentum, we might guess from the analogy to electromagnetism

$$\vec{\wp} = \frac{1}{c^2}\vec{j}_\varepsilon = -\frac{\partial\phi}{\partial t}\vec{\nabla}\phi$$

then

$$\frac{\partial \wp^i}{\partial t} = -\frac{\partial^2\phi}{\partial t^2}\vec{\nabla}^i\phi - \frac{\partial\phi}{\partial t}\nabla^i\frac{\partial\phi}{\partial t}$$

$$= -(c^2\nabla^2\phi)\nabla^i\phi - \tfrac{1}{2}\nabla^i\left(\frac{\partial\phi}{\partial t}\right)^2$$

$$= -c^2\nabla^j\left((\nabla^j\phi)(\nabla^i\phi)\right) + c^2\nabla^j\phi(\nabla^i\nabla^j\phi)$$
$$- \tfrac{1}{2}\nabla^i\left(\frac{\partial\phi}{\partial t}\right)^2$$

or

$$\frac{\partial \mathcal{P}^i}{\partial t} = -\nabla^j \left\{ c^2 \nabla^i\phi \nabla^j\phi + \frac{1}{2}\delta^{ij}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - c^2(\vec{\nabla}\phi)^2\right] \right\}$$

so that $\vec{\mathcal{P}}$ is also locally conserved

$$\frac{\partial \mathcal{P}^i}{\partial t} + \nabla^j T^{ji} = 0$$

with

$$\vec{\mathcal{P}} = -\frac{\partial\phi}{\partial t}\vec{\nabla}\phi$$

$$T^{ij} = c^2 \nabla^i\phi \nabla^j\phi + \delta^{ij}\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - c^2(\vec{\nabla}\phi)^2\right]$$

The total energy of the wave field is

$$E = \int d^3x \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}c^2(\vec{\nabla}\phi)^2 \right\}$$

This is a very general expression. Given any elastic medium with a degree of freedom $\phi(\vec{x})$ that can vibrate, the expansion of the expression for the energy for small vibrations is sure to produce a formula like this. There is kinetic energy in the motion of $\phi(\vec{x})$, and there is strain when $\phi(\vec{x})$ varies from place to place. It is the same story for vibrating strings, drum-heads, water waves, elastic solids, even the fluctuation from a uniform state of magnetism or other kinds of atomic order. In all cases, we find an energy function of the form:

$$E = \int d^3x \left\{ \frac{1}{2} \rho \left( \frac{\partial \chi}{\partial t} \right)^2 + \frac{1}{2} \kappa (\vec{\nabla} \chi)^2 \right\}$$

$$+ \mathcal{O}(\chi^3)$$

It follows from this expression that $\chi$ obeys the wave equation, with

$$c^2 = \sqrt{\frac{\kappa}{\rho}}$$