Electromagnetic Waves - 2

In the previous lecture, I analyzed an electromagnetic wave which had finite extent in z but infinite extent in x and y. Let's now examine what happens if the wave has finite extent in all directions (still with $a \gg \lambda$).

Start with the initial condition: At $t=0$,

$$\vec{E}(\vec{x}) = \mathrm{Re}\left\{ \int \frac{d^3k}{(2\pi)^3} \; \hat{x}_\perp \, E_0 \; e^{i\vec{k}\cdot\vec{x}} \; [2\pi a^2]^{3/2} \, e^{-\frac{1}{2}a^2(\vec{k}-\vec{k}_0)^2} \right\}$$

$$\vec{B}(\vec{x}) = \mathrm{Re}\left\{ \int \frac{d^3k}{(2\pi)^3} \; \hat{y}_\perp \frac{1}{c} E_0 \; e^{i\vec{k}\cdot\vec{x}} \; [2\pi a^2]^{3/2} \, e^{-\frac{1}{2}a^2(\vec{k}-\vec{k}_0)^2} \right\}$$

where $\vec{k}_0 = k_0 \hat{z}$. To be careful, we should write

$$\hat{x}_\perp = \hat{x} - \hat{k}\, \hat{k}\cdot\hat{x} \qquad \hat{y}_\perp = \hat{k} \times \hat{x}_\perp$$

But if $k_0 a \gg 1$, the corrections due to these modifications are very small ($O(\frac{1}{k_0 a})$) and can be neglected. To find the initial waveform, we need to do the integral

$$\int \frac{d^3k}{(2\pi)^3} \; e^{i\vec{k}\cdot\vec{x}} \; e^{-\frac{1}{2}a^2(\vec{k}-\vec{k}_0)^2}$$

$$= e^{i\vec{k}_0\cdot\vec{x}} \int \frac{d^3(\vec{k}-\vec{k}_0)}{(2\pi)^3} \; e^{i(\vec{k}-\vec{k}_0)\cdot\vec{x}} \; e^{-\frac{1}{2}a^2(\vec{k}-\vec{k}_0)^2}$$

This is just a set of three Gaussian integrals, and we can evaluate them in the familiar way, by completing the square in the exponent:

$$= e^{i\vec{k}_0\cdot\vec{x}} \int \frac{d^3(k-k_0)}{(2\pi)^3}\, e^{-\frac{1}{2}a^2(\vec{k}-\vec{k}_0 - i\frac{\vec{x}}{a^2})^2}\, e^{-\frac{1}{2a^2}x^2}$$

$$= e^{i\vec{k}_0\cdot\vec{x}}\, e^{-\frac{1}{2a^2}x^2}\, \frac{1}{(2\pi a^2)^{3/2}}$$

so the initial wave forms are

$$\vec{E} \cong \hat{x}\, E_0 \cos \vec{k}_0\cdot\vec{x}\; e^{-\frac{(\vec{x})^2}{2a^2}}$$

$$\vec{B} \cong \hat{y}\, E_0 \cos \vec{k}_0\cdot\vec{x}\; e^{-(\vec{x})^2/2a^2}$$

Now make these functions time-dependent. Let me use $\omega(\vec{k})$, a general function of $|\vec{k}|$, before specializing to the required form

$$\omega(\vec{k}) = c\,|\vec{k}|$$

We have

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left\{ \int \frac{d^3k}{(2\pi)^3}\, \hat{x}\, E_0\, e^{i\vec{k}\cdot\vec{x} - i\omega t}\, [2\pi a^2]^{3/2}\, e^{-\frac{1}{2}a^2(\vec{k}-\vec{k}_0)^2} \right\}$$

so to find the waveform at later times we need to compute

$$I = \int \frac{d^3k}{2\pi}\, e^{i\vec{k}\cdot\vec{x} - i\omega(\vec{k})t}\, e^{-\frac{1}{2}a^2(\vec{k}-\vec{k}_0)^2}$$

We would like to shift to the variable

$$\vec{\varkappa} = (\vec{k}-\vec{k}_0) = (k^x, k^y, k^z - k_0)$$

Expand $\omega(k)$ about $k = k_0$

$$\vec{k} = (\kappa^x, \kappa^y, k_0 + \kappa^z)$$

$$|\vec{k}| = \left[(\kappa^x)^2 + (\kappa^y)^2 + (k_0 + \kappa^z)^2 \right]^{\frac{1}{2}}$$

$$= k_0 + \kappa^z + \frac{1}{2k_0}\left[(\kappa^x)^2 + (\kappa^y)^2\right] + \mathcal{O}(\kappa^3)$$

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk}\Big|_0 \left(\kappa^z + \frac{1}{2k_0}\left[(\kappa^x)^2 + (\kappa^y)^2\right]\right)$$

$$+ \frac{1}{2}\frac{d^2\omega}{dk^2}\Big|_0 (\kappa^z)^2 + \mathcal{O}(\kappa^3)$$

where the derivatives of $\omega(k)$ are evaluated at $k = k_0$.

Then the integral on the previous page becomes:

$$\int \frac{d^3\kappa}{(2\pi)^3}\, e^{i\vec{k}_0\cdot\vec{x}}\, e^{i\vec{\kappa}\cdot\vec{x}}$$

$$\cdot\, e^{-i\left[\omega_0 + \frac{d\omega}{dk}\big|_0 \left(\kappa^z + \frac{(\kappa^x)^2 + (\kappa^y)^2}{2k_0}\right) + \frac{1}{2}\frac{d^2\omega}{dk^2}(\kappa^z)^2\right] t}$$

$$\cdot\, e^{-\frac{1}{2}a^2\kappa^2}$$

The factor $e^{i\vec{k}_0\cdot\vec{x} - i\omega_0 t}$ can be pulled out in front. Of the rest, the terms depending on κ^x are:

$$\int \frac{d\kappa^x}{2\pi}\, e^{i\kappa^x x}\, e^{-i\left[\frac{1}{2k_0}\frac{d\omega}{dk}\big|_0\right](\kappa^x)^2 t}\, e^{-\frac{1}{2}a^2(\kappa^x)^2}$$

completing the square, this becomes (with $\Delta = \frac{1}{k_0}\frac{d\omega}{dk}\big|_0$)

$$\int \frac{dk^x}{2\pi}\, e^{-\frac{1}{2}(a^2+i\Delta_x t)(k^x - i\frac{x}{a^2+i\Delta_x t})^2} \cdot e^{-\frac{1}{2}x^2/(a^2+i\Delta_x t)}$$

$$= \frac{1}{\sqrt{2\pi(a^2+i\Delta_x t)}} \exp\left[-\frac{1}{2}x^2\left[\frac{a^2-i\Delta_x t}{a^4+(\Delta_x t)^2}\right]\right]$$

The exponential has an oscillation, but its envelope is

$$e^{-x^2/2a_x^2(t)}$$

where

$$a_x(t) = \frac{[a^4+(\Delta_x t)^2]^{\frac{1}{2}}}{a} \cong a + \frac{1}{2}\frac{(\Delta_x t)^2}{a^3} + \cdots$$

$a_x(t)$ is the width of the wave in x. Apparently, the wavepacket spreads out slowly. For $\omega = ck$

$$a_x(t) = a\left(1 + \frac{1}{2}\frac{(ct)^2}{a^4 k_0^2} + \cdots\right)$$

so the packet spreads out only after it travels a distance

$$ct \sim a \cdot (k_0 a)$$

where typically $k_0 a \gg 1$.

The ~~integral~~ over k^y has the same behavior

But the integral over k^z shows some additional interesting behavior:

$$\int \frac{d\kappa^z}{2\pi} \; e^{i\kappa^z(z) - i\left(\frac{d\omega}{dk}\Big|_{K_0^z} t\right)} \; e^{-\frac{1}{2}(a^2 + i\Delta_z t)(\kappa^z)^2}$$

$$= \int \frac{d\kappa^z}{2\pi} \; e^{-\frac{1}{2}(a^2+i\Delta_z t)\left[\kappa^z - i\frac{(z - V_g t)}{(a^2 + i\Delta_z t)}\right]^2}$$

$$\cdot e^{-\frac{1}{2}\frac{(z-V_g t)^2}{a^2 + i\Delta_z t}}$$

where

$$V_g = \frac{d\omega}{dk}\Big|_0 \qquad \Delta_z = \frac{d^2\omega}{dk^2}\Big|_0$$

$$= \frac{1}{\sqrt{2\pi(a^2 + i\Delta_z t)}} \; \exp\left[-\frac{1}{2}\frac{(z - V_g t)^2 (a^2 - i\Delta_z t)}{[a^4 + (\Delta_z t)^2]}\right]$$

The envelope is

$$e^{-\frac{1}{2}\frac{(z - V_g t)^2}{(a_z(t))^2}}$$

where $a_z(t) = \dfrac{[a^4 + (\Delta_z t)^2]^{1/2}}{a}$ and $V_g = \dfrac{d\omega}{dk}\Big|_{k=0}$

The waveform spreads slowly as before. We also see it moving in the $\hat{z}$ direction. The velocity of the packet's motion is V_g, the "group velocity".

In all, the envelope of the wave packet is

$$e^{-\frac{1}{2}\frac{(x^2+y^2)}{a_\perp^2(t)}} \; e^{-\frac{1}{2}(z-V_g t)^2\frac{1}{a_z^2(t)}}$$

where $a_\perp(t) = \left[\frac{a^4+(\Delta_\perp t)^2}{a^2}\right]^{\frac{1}{2}}$ $\qquad a_z(t) = \left[\frac{a^4+(\Delta_z t)^2}{a^2}\right]^{\frac{1}{2}}$

$$\Delta_\perp = \frac{1}{k_0}\frac{d\omega}{dk}\Big|_0 \qquad \Delta_z = \frac{d^2\omega}{dk^2}\Big|_0$$

For an electromagnetic wave

$$\frac{\omega}{k} = \frac{d\omega}{dk} = c$$

so the whole wave moves with the velocity c. But this is an exception among waves found in Nature. More typically, we find that $\omega(k)$ is a more general function.

Then the wave oscillations have the form

$$e^{ikz-i\omega t}$$

so the crests and troughs of the wave move with velocity

$$v_p = \frac{\omega}{k} \qquad \text{the "phase velocity"}$$

while the envelopes of wave packets move with the velocity

$$v_g = \frac{d\omega}{dk} \qquad \text{the "group velocity"}$$

The energy and momentum in a wave move with the group velocity.
There are examples (e.g. water waves in shallow water) for which $\frac{d\omega}{dk} < 0$, so that the crests move one way but the energy moves in the opposite direction.

So far, we have discussed only the general spatial form of an electromagnetic wave, but the wave has one more degree of freedom, the <u>direction</u> in which it points. For given $\vec{k}$, $\vec{E}_0$ can point in any direction perpendicular to $\vec{k}$. In particular, there are two orthogonal choices

$$\vec{k} \parallel \hat{z} \Rightarrow \vec{E}_0 = \cos\alpha \, \hat{x} + \sin\alpha \, \hat{y}$$

The direction of $\vec{E}_0$, written as a unit vector $\vec{\epsilon}$, is called the <u>polarization</u> of the wave. For a wave with polarization $\vec{\epsilon}$, we'll write

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left\{ \vec{\epsilon}\, E_0 \, e^{+i\vec{k}\cdot\vec{x} - i\omega t} \right\}$$

$$\vec{B}(x,t) = \mathrm{Re}\left\{ \hat{k}\times\vec{\epsilon}\, \frac{E_0}{c} \, e^{i\vec{k}\cdot\vec{x} - i\omega t} \right\}$$

with $|\vec{\epsilon}|^2 = 1$, the energy density in the wave is

$$\langle \mathcal{E} \rangle = \frac{1}{2} \epsilon_0 \mathbb{E}_0^2$$

It may seem to you that a polarization vector should always be real, but there are important examples of polarization vectors that are complex. Consider, for $\vec{k} = k\hat{z}$

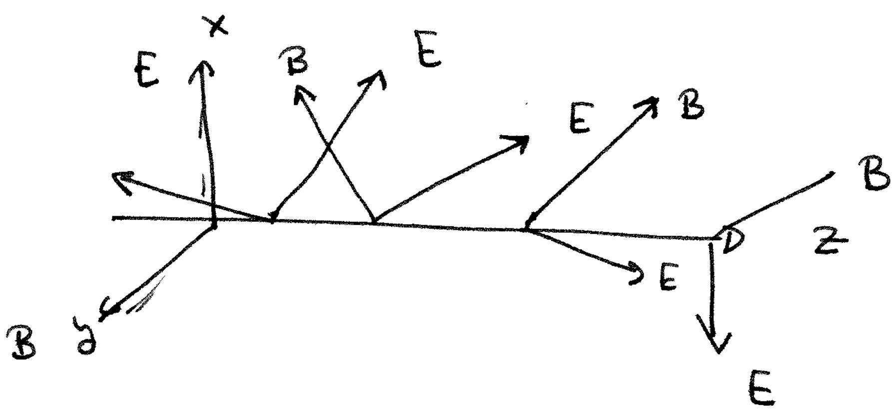
$$\vec{\epsilon} = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{y})$$

This gives rise to

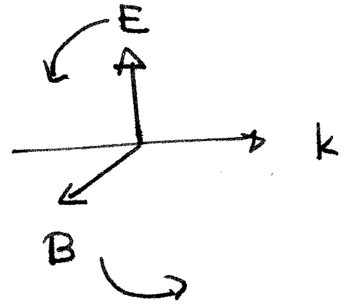
$$\vec{E} = \mathbb{E}_0 \cdot \frac{1}{\sqrt{2}} (\hat{x} \cos(kx-\omega t) - \hat{y} \sin(kx-\omega t))$$

$$\vec{B} = \frac{1}{c}\mathbb{E}_0 \frac{1}{\sqrt{2}} (\hat{y} \cos(kx-\omega t) + \hat{x} \sin(kx-\omega t))$$

at fixed time, this wave looks like:



The direction of $\vec{E}$ and $\vec{B}$ rotate about the axis given by $\vec{k}$, always remaining perpendicular. As the wave moves, the values of $\vec{E}$ at a given point rotate in the right-hand sense about the $\hat{k}$ axis as a function of time



This polarization state is called "right-handed circular polarization". The opposite state, given by

$$\vec{\epsilon} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$$

is "left-handed circular polarization".

A circularly polarized wave is equivalent to the superposition of a wave w. $\vec{\epsilon} = \hat{x}$ and a wave with $\vec{\epsilon} = \hat{y}$, 90° out of phase. To create a circularly polarized wave, one might begin with a wave of polarization $\vec{\epsilon} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ and use an anisotropic ("birefringent") material to introduce a delay of the $\hat{y}$ component relative to the $\hat{x}$ component. A device that introduces the required 90° phase delay is called a "quarter-wave plate".

In quantum mechanics, circular polarization has a special significance. The quanta of the electromagnetic wave - photons - with right (left) handed circular polarization carry definite angular momentum

$$L_z = +\hbar \quad (-\hbar)$$

about the z-axis identified with the direction of motion ($\hat{k}$).

The energy density of the circularly polarized wave is

$$\mathcal{E} = \tfrac{1}{2}\epsilon_0 (\vec{E})^2 + \tfrac{1}{2\mu_0}(\vec{B})^2$$

$$= \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2 \left[\tfrac{1}{2}\cos^2(kx-\omega t) + \tfrac{1}{2}\sin(kx-\omega t)\right]$$
$$+ \tfrac{1}{2\mu_0}\tfrac{1}{c^2}\mathbb{E}_0^2 \left[\tfrac{1}{2}\cos^2(kx-\omega t) + \tfrac{1}{2}\sin(kx-\omega t)\right]$$

$$= \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2$$

in general, we will have

$$\langle \mathcal{E} \rangle = \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2$$

if the waveform has the form given on p. 7 with

$$\vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^* = 1$$