Electromagnetic Waves

Now that we have the basic equations for electromagnetism, let's look for solutions to them. I'll begin with solutions in empty space, with no charged particles around. Since Maxwell's equations are linear differential equations with constant coefficients, we should expect to find solutions which are exponentials or oscillations. In our study of circuits, we represented a true oscillatory solution as

$$I(t) = |I_0| \cos(\omega t + \phi) = \mathrm{Re}\, I_0 e^{-i\omega t}$$

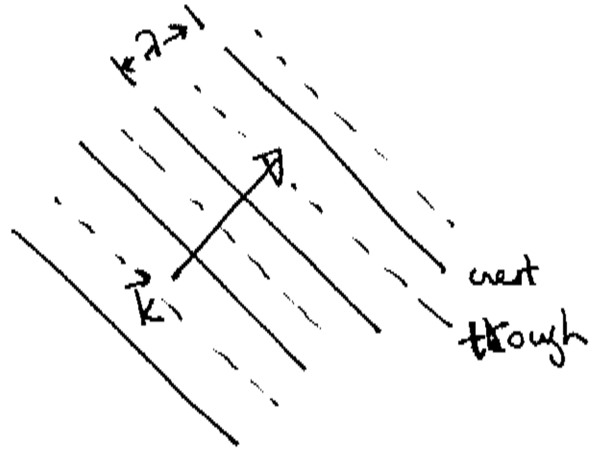
ω is the frequency of the oscillation, and $T = 2\pi/\omega$ is the period. Similarly, we can represent an oscillation in space as

$$f(x) = \mathrm{Re}\, f_0 e^{ikx} = |f_0| \cos(kx + \alpha)$$

[for $f_0 = |f_0| e^{i\alpha}$]. The parameter k is the wave number, related to the wavelength by $\lambda = 2\pi/k$. The function $f(x)$ can also be thought of as a wave in 3-dimensional space which is constant in y and z. A more general wave would be

$$f(\vec{x}) = \mathrm{Re}\, f_0 e^{i\vec{k}\cdot\vec{x}} = |f_0| \cos(\vec{k}\cdot\vec{x} + \alpha)$$

The wave varies most rapidly in the direction $\vec{k}$, with $\lambda = 2\pi/|\vec{k}|$, and it is constant in directions orthogonal to $\vec{k}$:



I would like to look for solutions of Maxwell's equations that vary both in space and in time:

$$\vec{E} = \mathrm{Re}\{\vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}\}$$

$$\vec{B} = \mathrm{Re}\{\vec{B}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}\}$$

where $\vec{E}_0, \vec{B}_0$ are constant complex vectors.

As we did in the circuit problems, extend Maxwell's equations to complex variables and plug in the above expressions. We find $(\rho = \vec{j} = 0)$:

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \Rightarrow \quad +i\vec{k}\cdot\vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad +i\vec{k}\cdot\vec{B}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \left[i\vec{k}\times\vec{E}_0 - i\omega\vec{B}_0\right] e^{-i\omega t + i\vec{k}\cdot\vec{x}} = 0$$

$$\vec{\nabla}\times\vec{B} - \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0 \Rightarrow \left[i\vec{k}\times\vec{B}_0 + i\omega\mu_0\epsilon_0\vec{E}_0\right] e^{-i\omega t + i\vec{k}\cdot\vec{x}} = 0$$

so we have a solution if the following constraints are satisfied:

$$\vec{k}\cdot\vec{E}_0 = 0 \qquad\qquad \vec{k}\cdot\vec{B}_0 = 0$$

$$\vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0 \qquad\qquad \vec{E}_0 = -\frac{1}{\mu_0\epsilon_0}\frac{1}{\omega}\vec{k}\times\vec{B}_0$$

Notice that, if $\vec{B}_0 = \vec{k}\times(\vec{V})$, then automatically $\vec{k}\cdot\vec{B}_0 = 0$, and similarly for $\vec{E}_0$. But there is one more consistency condition that must be satisfied:

$$\vec{B}_0 = \frac{1}{\omega}\vec{k}\times\left(-\frac{1}{\mu_0\epsilon_0}\frac{1}{\omega}\vec{k}\times\vec{B}_0\right)$$

$$= -\frac{1}{\mu_0\epsilon_0}\frac{1}{\omega^2}\vec{k}\,\underbrace{(\vec{k}\cdot\vec{B}_0)}_{=0\,!} + \frac{1}{\mu_0\epsilon_0}\frac{1}{\omega^2}\vec{B}_0(\vec{k}\cdot\vec{k})$$

or $$\vec{B}_0 = \left[\frac{1}{\mu_0\epsilon_0}\frac{k^2}{\omega^2}\right]\vec{B}_0$$

So, the most general solution of this oscillatory form is given by:

(1) $\vec{E}_0$ is a general vector $\perp$ to $\vec{k}$

(2) $\vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0$ (also $\perp$ to $\vec{k}$)

(3) $\omega = \frac{1}{\sqrt{\epsilon_0\mu_0}}|\vec{k}|$

Let me abbreviate:

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$$

Then the last two conditions read:

(2') $\vec{B}_0 = \frac{1}{c}\hat{k}\times\vec{E}_0$

(3') $\omega = c|\vec{k}|$

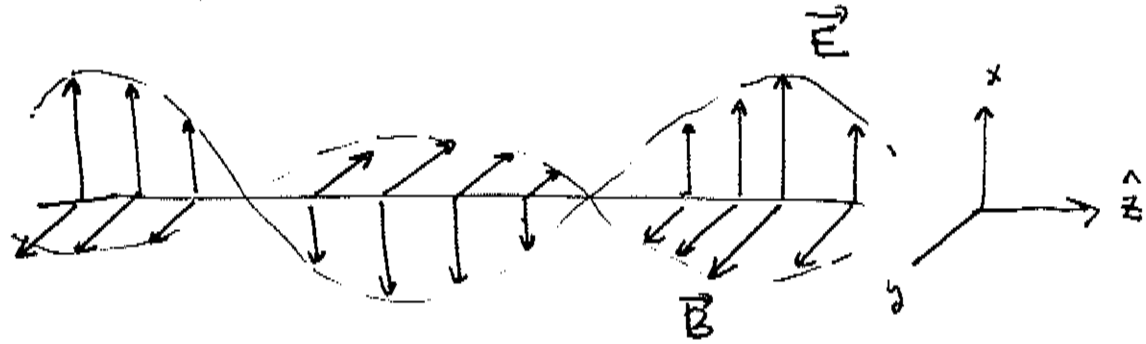
To take a particular example, let

$$\vec{k}\parallel\hat{z} \qquad \vec{E}_0 = \hat{x}\underbrace{\mathbb{E}_0}_{\text{real.}} \qquad \vec{B}_0 = \hat{y}\frac{1}{c}\mathbb{E}_0$$

then

$$\vec{E} = \hat{x}\,\mathbb{E}_0\cos(\omega t - \vec{k}\cdot\vec{x})$$

$$\vec{B} = \hat{y}\left(\tfrac{1}{c}\right)\mathbb{E}_0\cos(\omega t - \vec{k}\cdot\vec{x})$$



The crests move in such a way that

$$\omega t - \vec{k}\cdot\vec{x} = (\text{const})$$

$$\omega t - kz = (\text{const})$$

$$z = \frac{\omega}{k}t + (\text{const})$$

so! the crests of the wave move in the direction of $\vec{k}$ with velocity $v = c$.

From the equation $c = \frac{\omega}{|\vec{k}|}$, it is clear that c has the units m/sec. This is less clear from the relation

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

but

$$\epsilon_0 \sim \frac{q^2}{N\,m^2} \qquad \mu_0 \sim \frac{N}{A^2} = \frac{N}{C^2/\text{sec}^2}$$

so indeed

$$\frac{1}{\sqrt{\epsilon_0\mu_0}} = \left(\frac{Nm^2}{C^2}\cdot\frac{C^2}{N\,\text{sec}^2}\right)^{1/2} = \text{m/sec}$$

Actually, since we know ϵ_0 and μ_0 from electro- and magnetostatics, we can get the number:

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = \left[\frac{1}{(8.85\times10^{-12}\ C^2/Nm^2)\cdot(4\pi\times10^{-7}\ N/A^2)}\right]^{1/2}$$

$$= 3.00\times10^{8}\ \text{m/sec}$$

which is the speed of light. For the rest of the course, we will accumulate evidence that these waves are light, and other forms of radiation found in Nature. It is amazing that Maxwell's equations, which we set up to describe the forces due to charges and currents, also describe forms of radiation that are completely liberated from charges and currents and move freely through empty space.

The wave solution on p.4 actually extends infinitely in all directions. To get a better feeling for electromagnetic waves, it will be useful to consider solutions that are more localized. I will first consider solutions localized in z, then solutions localized in all three dimensions.

Consider a $\vec{k}$ parallel to z. Then, for any k, we have a solution to Maxwell's equations:

$$\vec{E} = \mathrm{Re}\left[\hat{x} E_0 \, e^{-i\omega t + ikz}\right] \qquad \vec{B} = \mathrm{Re}\left[\hat{y} c E_0 \, e^{-i\omega t + ikz}\right].$$

any linear combination of these solutions is also a solution. So, form a linear combination of the form of a Gaussian wavepacket: At $\underline{t=0}$ $\qquad k_0 \geq 0$?

$$\vec{E} = \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \hat{x} \, E_0 \, e^{ikz} \sqrt{2\pi a^2} \, e^{-\frac{1}{2}a^2(k-k_0)^2} \right\}$$

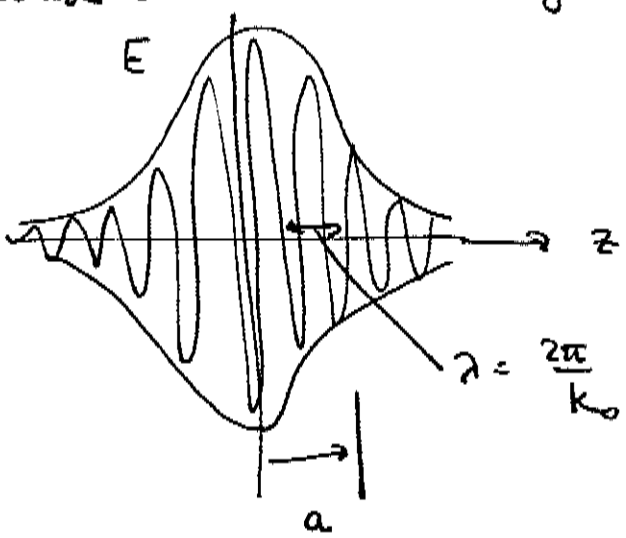
$$\vec{B} = \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \hat{y}\left(\tfrac{1}{c}\right) E_0 \, e^{ikz} \sqrt{2\pi a^2} \, e^{-\frac{1}{2}a^2(k-k_0)^2} \right\}$$

(The factors $\sqrt{2\pi a^2}$ are added for later convenience.) To interpret this waveform, do the integral

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikz}\, e^{-\frac{1}{2}a^2(k-k_0)^2} = \frac{1}{\sqrt{2\pi a^2}}\, e^{ik_0 z}\, e^{-\frac{1}{2a^2}z^2}$$

so the waveform at $t=0$ is (for E_0 real)

$$\vec{E} = \hat{x}\, E_0 \cos k_0 z\, e^{-\frac{z^2}{2a^2}} \qquad \vec{B} = \hat{y}\left(\tfrac{1}{c}\right)E_0 \cos k_0 z\, e^{-z^2/2a^2}$$



This is a pulse of field of length a, containing an oscillation with wavelength $\lambda = 2\pi/k_0$. From here on, it will be useful to think about the limit

$$a \gg \lambda \qquad \text{or} \qquad k_0 a \gg 1$$

so that the pulse contains a large number of wavelengths. This is a typical situation for emitters of electromagnetic radiation from lightbulbs and lasers to radio antennae.

Maxwell's equations are first-order differential equations. So, given an initial condition, we should be able to integrate them in time to find a unique solution. Actually, we know this solution already; we only need to restore the dependence on t in the formulae on p.6. Remembering that $\omega = ck$

$$\vec{E}(z,t) = \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ \hat{x}\, E_0\ e^{ikz - ickt} \sqrt{2\pi a^2}\ e^{-\frac{1}{2}a^2(k-k_0)^2} \right\}$$

again it is easy to do the integral

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{ikz - ickt}\ e^{-\frac{1}{2}(k-k_0)^2}$$

$$= e^{ik_0(z-ct)} \int_{-\infty}^{\infty} \frac{d(k-k_0)}{2\pi}\ e^{i(k-k_0)(z-ct)}\ e^{-\frac{1}{2}(k-k_0)^2}$$

$$= e^{ik_0(z-ct)}\ e^{-(z-ct)^2/2a^2} \frac{1}{\sqrt{2\pi a^2}}$$

so we find

$$\vec{E}(z,t) = \hat{x}\, E_0 \cos(k_0(z-ct))\ e^{-(z-ct)^2/2a^2}$$

$$\vec{B}(z,t) = \hat{y}\frac{1}{c}E_0 \cos k_0(z-ct)\ e^{-(z-ct)^2/2a^2}$$

The pulse keeps its form and moves along so that its center is at

$$(z - ct) = 0$$

$$z = ct$$

so we have a pulse of wave moving in the z direction at the speed of light.

Since we know that electric and magnetic fields have energy and momentum, we might suspect that these waves are carrying energy and momentum in the $\hat{z}$ direction. So, let's compute the total energy and momentum of the wave packet. The packet has length of order a. It is very large in the x and y directions; let's say for the moment that it has finite area A. If $a \gg \lambda$, it makes sense to average functions that oscillate rapidly over several wavelengths. I'll denote this average by $\langle\,\rangle$.

Consider first the energy of the wave. This is given by

$$E = \int d^3x \left\{ \frac{1}{2}\epsilon_0 (\vec{E})^2 + \frac{1}{2\mu_0} (\vec{B})^2 \right\}$$

the $(\vec{E})^2$ term is

$$\frac{1}{2}\epsilon_0 (\vec{E})^2 = \frac{1}{2}\epsilon_0 \left(E_0^2 \cos^2 k_0(z-ct)\, e^{-(z-ct)^2/a^2} \right)$$

If $a \gg 2\pi/k_0$, we can average over a distance b

$$a \gg b \gg \lambda$$

$$\left\langle \frac{1}{2}\epsilon_0 (\vec{E})^2 \right\rangle = \frac{1}{2}\epsilon_0 \cdot \frac{1}{2} E_0^2\, e^{-(z-ct)^2/a^2}$$

similarly

$$\left\langle \frac{1}{2}\frac{1}{\mu_0} (\vec{B})^2 \right\rangle = \underbrace{\frac{1}{2\mu_0} \frac{1}{c^2}}_{=\frac{1}{2}\epsilon_0} E_0^2 \cdot \frac{1}{2} \cdot e^{-(z-ct)^2/a^2}$$

These two terms are equal, like the kinetic and potential energy of a harmonic oscillator, and so the total energy in the wave is:

$$E = \int d^3x \;\; \tfrac{1}{2}\epsilon_0 \, \mathbb{E}_0^2 \, e^{-(z-ct)^2/a^2}$$

$$= A \cdot \sqrt{\pi}\, a \cdot \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2$$

The momentum in the wave is

$$\vec{P} = \int d^3x \;\; \epsilon_0 \; \vec{E} \times \vec{B}$$

$$= \int d^3x \;\; \epsilon_0 \frac{1}{c} \mathbb{E}_0^2 \cdot \hat{z} \, \cos^2 k_0(z-\omega t) \;\; e^{-(z-ct)^2/a^2}$$

averaging as above

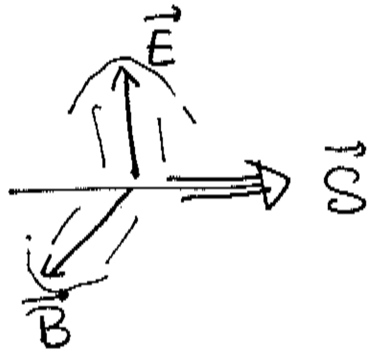
$$= \int d^3x \;\; \frac{\epsilon_0}{c} \mathbb{E}_0^2 \cdot \frac{1}{2} \cdot e^{-(z-ct)^2/a^2} \; \hat{z}$$

$$= A \cdot \sqrt{\pi}\, a \cdot \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2 \; \frac{1}{c} \; \hat{z}$$

or

$$\vec{P} = E/c \;\; \hat{k}$$

The direction of the momentum ~~[illegible]~~ is given by the direction of the Poynting vector



Notice that the factor of $\frac{1}{2}$ from averaging over the oscillation is one that we met earlier in our study of oscillating circuits:

$$\langle I(t)^2 \rangle = \langle (\mathrm{Re}\, I_0 e^{-i\omega t})^2 \rangle = \frac{1}{2} |I_0|^2$$

You should expect this factor to appear in any energy calculation involving oscillating systems.

We can also compute the fluxes of energy and momentum.

$$\vec{j}_\mathcal{E} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$= \frac{1}{\mu_0 c} \hat{z}\, \mathbb{E}_0^2 \cos^2 k_0(z-ct)\, e^{-(z-ct)^2/a^2}$$

$$\langle \vec{j}_\mathcal{E} \rangle = \frac{1}{2} \epsilon_0 c\, \hat{z}\, \mathbb{E}_0^2\, e^{-(z-ct)^2/a^2}$$

so

$$\vec{j}_\mathcal{E} = c\hat{z} \cdot \mathcal{E} = (\overrightarrow{\text{velocity}}) \cdot (\text{energy density})$$

The energy flux through a surface normal to $\hat{z}$ at z is

$$\Phi_\mathcal{E}(z,t) = \int d^2x\ \hat{z} \cdot \vec{j}_\mathcal{E} = \frac{1}{2}\epsilon_0 c\, \mathbb{E}_0^2\, e^{-(z-ct)^2/a^2} \cdot A$$

$$= \text{J/sec.}$$

If we integrate this from $t=-\infty$ to $t=\infty$, the entire energy of the wave should pass through the surface. Let's check this:

$$\int_{-\infty}^{\infty} dt \; \mathcal{E}_z(z,t) = \int_{-\infty}^{\infty} dt \cdot c \; \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2 \; e^{-(z-ct)^2/a^2} \cdot A$$

$$= A \cdot \sqrt{\pi}\, a \cdot \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2 = E \checkmark$$

The momentum current is

$$T^{ij} = +\left[\tfrac{1}{2}\epsilon_0 (\vec{E})^2 + \tfrac{1}{2\mu_0}(\vec{B})^2\right]\delta^{ij} - \epsilon_0 E^i E^j - \tfrac{1}{\mu_0} B^i B^j$$

$$= \Big\{\Big[\tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2 \cos^2 k_0(z-ct) + \tfrac{1}{2\mu_0}\tfrac{1}{c^2}\mathbb{E}_0^2 \cos^2 k_0(z-ct)\Big]\delta^{ij}$$

$$- \epsilon_0 \mathbb{E}_0^2 \, \hat{x}^i \hat{x}^j \cos^2 k(z-ct) - \tfrac{1}{\mu_0}\tfrac{1}{c^2}\mathbb{E}_0^2 \hat{y}^i \hat{y}^j \cos^2 k(z-ct)\Big\}$$

$$\cdot \, e^{-(z-ct)^2/a^2}$$

$$\langle T^{ij}\rangle = \Big\{\Big(\tfrac{1}{2}\epsilon_0 \cdot \tfrac{1}{2}\mathbb{E}_0^2 + \tfrac{1}{2\mu_0}\epsilon\mu_0 \tfrac{1}{2}\mathbb{E}_0^2\Big)\delta^{ij}$$

$$- \epsilon_0 \mathbb{E}_0^2 \cdot \tfrac{1}{2}\, \hat{x}^i \hat{x}^j - \frac{\epsilon_0\mu_0}{\mu_0}\mathbb{E}_0^2 \hat{y}^i \hat{y}^j \cdot \tfrac{1}{2}\Big\}$$

$$\cdot \, e^{-(z-ct)^2/a^2}$$

$$= \tfrac{1}{2}\epsilon_0 \mathbb{E}_0^2 e^{-(z-ct)^2/a^2} \cdot \left[\delta^{ij} - \hat{x}^i\hat{x}^j - \hat{y}^i\hat{y}^j\right]$$

the quantity in brackets is the matrix

$$[\delta^{ij} - \hat{x}^i\hat{x}^j - \hat{y}^i\hat{y}^j] = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\end{pmatrix}_{ij} - \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\end{pmatrix}_{ij} - \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0\end{pmatrix}_{ij}$$

$$= \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1\end{pmatrix}_{ij}$$

so actually T^{ij} has only one nonzero entry:

$$T^{zz} = \tfrac{1}{2}\epsilon_0 E_0^2\, e^{-(z-ct)^2/a^2}$$

$$= c\, \mathcal{P}^z$$

so z-momentum is transported in the z direction at speed c!

As for the energy

$$(\vec{\Phi}_P)^i = \int d^2x\ z^j\, T^{ji}$$

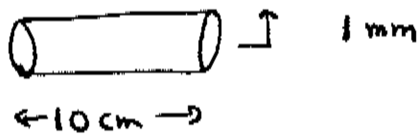
$$= \hat{z}^i \tfrac{1}{2}\epsilon_0 E_0^2\, e^{-(z-ct)^2/a^2}\cdot(\text{Area})$$

$$\int_{-\infty}^{\infty} dt\ \vec{\Phi}_P(z,t) = \hat{z}\ \frac{1}{c}\ \tfrac{1}{2}\epsilon_0 E_0^2\ A\cdot\sqrt{\pi}\,a$$

$$= \vec{P}$$

so the formal expressions from the last lecture allow us to visualize how energy and momentum are transported by the wave.

Just to get a feeling for orders of magnitude, a light pulse from a laser might contain 1 J in a volume



For red light, $\lambda \sim 6000$ Å $= 6 \times 10^{-7}$ m, so $\lambda \ll a$

The energy density

$$\mathcal{E} \sim 3 \times 10^{6} \ \mathrm{J/m^3}$$

Now, this is $\approx \frac{1}{2}\epsilon_0 E_0^2$

so, using $\epsilon_0 = 8. \times 10^{-12} \ \mathrm{C^2/Nm^2}$

$\mathrm{J = Nm}$

we have

$$E_0^2 \cong 7 \times 10^{17} \ \frac{\mathrm{J}}{\mathrm{m^3}} \cdot \frac{\mathrm{Nm^2}}{\mathrm{C^2}}$$

$$\cong 10^{18} \ \frac{\mathrm{J^2}}{\mathrm{C^2 m^2}}$$

$$E_0 \cong 10^{9} \ \frac{\mathrm{V}}{\mathrm{m}} \cong 0.1 \ \frac{\mathrm{V}}{\text{Å}}$$

(atomic scale)