The Stress Tensor

Jan. 29

In the previous lecture, I showed that Maxwell's equations respect the conservation of energy and momentum, with the following expressions for the energy and momentum of the electromagnetic field.

$$E = \int d^3x \left\{ \frac{1}{2}\epsilon_0 (\vec{E})^2 + \frac{1}{2\mu_0}(\vec{B})^2 \right\}$$

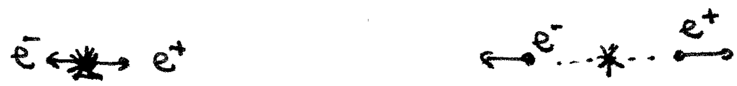
$$\vec{P} = \int d^3x \left\{ \epsilon_0 \vec{E} \times \vec{B} \right\}$$

However, in a theory of fields, we can expect more from a conservation law. Let me now explain what structure we should look for, and then we'll see if it is there.

For electric charge, we could also have been satisfied with a global conservation law. This would be

$$\frac{d}{dt} Q = 0 \qquad Q = \int d^3x \, \rho$$

However, electric charge is carried by electrons and protons, and these do not disappear in one place and reappear suddenly in another place. Rather, they move continuously from one place to another. In high-energy processes, electrons and anti-electrons (positrons) can be created in pairs, but the pair always emerges from a point and moves apart in a continuous way:



So, not only is the total electric charge in Nature conserved, but in any small region the charge changes only by charges flowing in or out of that region. The "continuity equation"

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{j} = 0$$

makes this precise: In any small volume, the change in the electric charge enclosed is accounted for by a flow through the walls.

Now, what about energy and momentum? In Newton's world, with instantaneous action at a distance, energy and momentum can disappear from one particle and reappear on another particle some distance away:

However, in the world of Faraday and Maxwell, the picture is quite different. A charged particle acting as a source of

$\vec{E}$ or $\vec{B}$ creates electromagnetic field. This field then pushes on another charged particle. Each influence of a particle on the field or of a field on a particle is local. This suggests ~~that~~ energy and momentum should actually be carried from place to place by the electromagnetic field. They should flow. A decrease of energy or momentum in a certain volume should be explained by a flow of energy or momentum out of that volume.

Is this in fact a property of Maxwell's equations? Let's first convert this idea into a mathematical statement, then test it. For simplicity, consider first the energy. Let $\mathcal{E}(\vec{x},t)$ be the density of energy at $\vec{x}$ at time t (measured in (J/m^3)). We will also need to speak about a current of energy. Let $\vec{J}_{\mathcal{E}}(\vec{x},t)$ be this current, with units of $J/m^2\,\mathrm{sec}$, such that

$$\Phi_{\mathcal{E}} = \int_S d^2x \; \hat{n}\cdot\vec{J}_{\mathcal{E}} = \text{"flux of energy"}$$

is the amount of energy flowing across the surface S per unit time. Then energy is conserved in any small region if the change in energy in that region is explained by the flow of energy in or out. This requires

$$\frac{\partial}{\partial t}\mathcal{E} + \vec{\nabla}\cdot\vec{j}_{\mathcal{E}} = 0$$

If such an equation holds everywhere, we say that energy is <u>locally conserved</u>.

Does such an equation hold in Maxwell's theory? Let's first look at the theory away from all charged particles. We expect that

$$\mathcal{E} = \frac{1}{2}\epsilon_0(\vec{E})^2 + \frac{1}{2\mu_0}(\vec{B})^2$$

since we saw in the last lecture that $E = \int d^3x\, \mathcal{E}$.

We also computed $\frac{\partial\mathcal{E}}{\partial t}$ and used Maxwell's equations to show that

$$\frac{\partial\mathcal{E}}{\partial t} = -\vec{\nabla}\left(\frac{1}{\mu_0}\vec{E}\times\vec{B}\right) - \vec{E}\cdot\vec{j}$$

The second term is the energy loss to matter — a local process. But we see that in the absence of matter this equation has the desired form

$$\frac{\partial\mathcal{E}}{\partial t} + \vec{\nabla}\cdot\vec{j}_{\mathcal{E}} = 0$$

where $\vec{j}_{\mathcal{E}} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

This quantity is canonically called $\vec{S}$ or "the Poynting vector".

If we now add matter back, we must ask whether the theory of the matter conserves energy locally. If so

$$\frac{\partial \mathcal{E}_m}{\partial t} + \vec{\nabla}\cdot\vec{j}_{\mathcal{E}m} = 0$$

in the absence of electric and magnetic fields. In a gas of particles of fixed density in which all particles have the same kinetic energy k and the same velocity $\vec{v}$

$$\mathcal{E}_m = \rho k \qquad \vec{j}_{\mathcal{E}m} = \rho \vec{v} k$$

When we have both particles and electromagnetic fields

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{E}\cdot\vec{j}$$

$$\frac{\partial \mathcal{E}_m}{\partial t} + \vec{\nabla}\cdot\vec{j}_{\mathcal{E}m} = +\vec{E}\cdot\vec{j}$$

and

$$\frac{\partial}{\partial t}(\mathcal{E} + \mathcal{E}_m) + \vec{\nabla}\cdot(\vec{S} + \vec{j}_{\mathcal{E}m}) = 0$$

Now I would like to describe the corresponding story for <u>momentum</u>. The basic principles are the same, but because momentum is a <u>vector</u> quantity, more indices appear. Let

$$\mathcal{P}^i(\vec{x},t) = \text{density of } i \text{ component of momentum at } (\vec{x},t)$$

so that

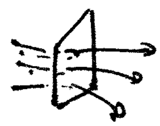
$$\vec{P} = \int d^3x\ \vec{\mathcal{P}}(x,t)$$

Local conservation of momentum is the statement

$$\frac{\partial}{\partial t} \mathcal{P}^i + \nabla^j T^{ji} = 0$$

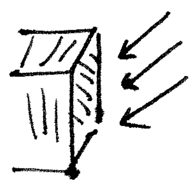
The quantity T^{ji} is the j^{th} component of the current of the i^{th} component of momentum. In other words

$$\Phi^i_S = \int_S d^2x \; \hat{n}^j \, T^{ji}$$



is the amount of the i^{th} component of momentum being transported across the surface S in a unit time.

A related situation is the following: Let S be a surface — a wall — in space, and let $\hat{n}$ be the outward normal. If the wall is absorbing momentum, the



body of which the wall is the boundary experiences a force. The force on the body is

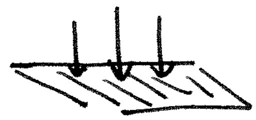
$$\frac{d\vec{P}}{dt} = - \int d^2x \, (\hat{n})^j \, T^{ji} = \left(\begin{array}{c} \text{momentum transported in} \\ \text{per unit time} \end{array} \right)$$

It is conventional to write this equation as

$$\frac{d\vec{P}}{dt} = + \int d^2x \; \hat{n}^j \, \sigma^{ji}$$

where σ^{ji} is called the <u>stress tensor</u>. In words,

σ^{ji} is the force in the i^{th} direction per unit area acting on a surface with outward normal in the j^{th} direction. The diagonal components of σ^{ji} represent forces acting normal to the surface; they are called <u>normal</u> stresses



For example, an ideal fluid has a stress tensor

$$\sigma^{ji} = -p\,\delta^{ij}$$

where p is the pressure, exerted inward on a surface. The off diagonal elements of σ^{ji} represent tangential forces

or <u>shear</u> <u>stresses</u>.

[Note that both <u>Griffiths</u> and <u>Heald</u> and <u>Marion</u> call the stress tensor T^{ji}. Thus, it is the <u>opposite</u> of what I call T^{ji}. <u>Jackson</u> uses the same convention as me for T^{ji}, but he calls it the stress tensor. Please be careful of varying conventions!]

Let's see if the electromagnetic field, described by Maxwell's equations, respects local conservation of momentum. In the previous lecture, I showed that the conserved total momentum of the electromagnetic field has the form

$$\vec{P} = \int d^3x \, \vec{\mathcal{P}} \qquad \text{where} \qquad \vec{\mathcal{P}} = \epsilon_0 \vec{E} \times \vec{B}$$

We computed $\frac{\partial \vec{\mathcal{P}}}{\partial t}$ and found the expression

$$\frac{\partial \mathcal{P}^i}{\partial t} = -\left[\rho E^i + (\vec{j} \times \vec{B})^i\right] + \nabla^j (E^i E^j) - \tfrac{1}{2} \nabla^i (\vec{E})^2$$
$$+ \nabla^j (B^i B^j) - \tfrac{1}{2} \nabla^i (\vec{B})^2$$

The first term is the momentum loss to matter through the electric and magnetic forces. This is local and is treated with the $(-\vec{E}\vec{j})$ term on p. 5. Away from charges, we have:

$$\frac{\partial \mathcal{P}^i}{\partial t} = \nabla^j \left(E^j E^i - \tfrac{1}{2} \delta^{ij} (\vec{E})^2 + B^j B^i - \tfrac{1}{2} (\vec{B})^2 \right)$$

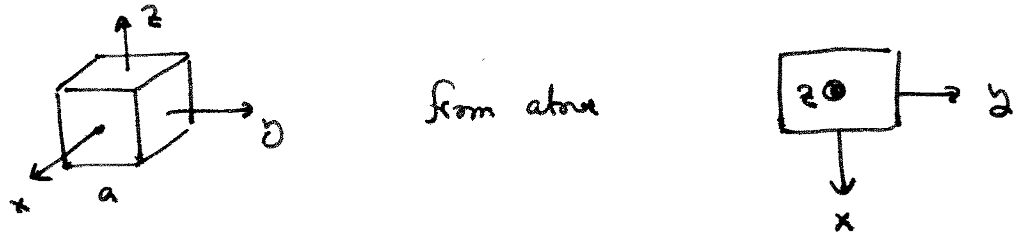
so the equation of local conservation is satisfied, with

$$\sigma^{ji} = -T^{ji} = -E^j E^i - B^j B^i + \tfrac{1}{2} \delta^{ij} \left[(\vec{E})^2 + (\vec{B})^2 \right]$$

A wonderful property of this expression is that it is a symmetric tensor:

$$\sigma^{ji} = \sigma^{ij}$$

The symmetry of the stress tensor is a reflection of very deep connections between energy and momentum and space-time geometry. However, It is easy to see that we would be in serious trouble if this relation did not hold. Consider a small cube of side a oriented so that the sides line up with the x, y, z axes:



If $\sigma^{xy} \neq 0$, the lower face in the right-hand picture experiences a force

$$\vec{f} = a^2 \sigma^{xy} \hat{y}$$

which leads to a torque about the center of the cube

$$\vec{\tau} = \hat{z} \cdot \left(a^2 \sigma^{xy} \cdot \frac{a}{2}\right)$$

Similarly, the stress on the top face gives a force

$$\vec{f} = -a^2 \sigma^{xy} \hat{y} \qquad \text{(since the normal points in the } -\hat{x} \text{ direction)}$$

and so also contributes $\vec{\tau} = +\hat{z}\left(a^2 \sigma^{xy} \cdot \frac{a}{2}\right)$

The stress on the left and right surfaces give forces in the x direction proportional to σ^{yx}, giving a torque with the opposite sign.

(right hand surface:) $\vec{f} = a^2 \sigma^{yx} \hat{x}$ $\qquad \vec{\tau} = -a^2 \sigma^{yx} \frac{a}{2} \hat{z}$

in all

$$\vec{\tau} = a^3 (\sigma^{xy} - \sigma^{yx}) \hat{z}$$

But, the moment of inertia of a small cube goes as

$$I \sim (\text{volume}) \cdot a^2 \sim a^5 \qquad \text{as } a \to 0$$

thus, the smaller the cube, the faster it is made to spin — a clearly unphysical result — unless

$$\sigma^{ij} = \sigma^{ji} \qquad \text{for all } j, i$$

The four elements $\mathcal{E}$, $\vec{j}_\mathcal{E}$, $\vec{\mathcal{P}}$, T^{ij} are actually components of a larger 4x4 tensor called the energy-momentum tensor $T^{\mu\nu}$ $\quad \mu, \nu = t, x, y, z$.

Roughly, we have

$$T^{tt} \cong \mathcal{E} \qquad T^{ti} \cong \mathcal{P}^i \qquad T^{it} \cong j_\mathcal{E}^i$$

Notice that

$$\vec{j}_\mathcal{E} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \vec{S}$$

$$\vec{\mathcal{P}} = \epsilon_0 \vec{E} \times \vec{B} = \epsilon_0 \mu_0 \vec{S}$$

The fact that these objects have the ~~same~~ form is a manifestation of the symmetry of the energy-momentum tensor. I will make this statement precise later in the course.

Before we finish the final study of conservation laws, I should say something about angular momentum. The angular momentum of the electromagnetic field is

$$L = \int d^3x \; \epsilon_0 \, \vec{r} \times (\vec{E} \times \vec{B}) = \int d^3x \; \vec{r} \times \vec{\mathcal{P}}$$

Let's check that this is conserved:

$$\frac{d\vec{L}}{dt} = \int d^3x \; \vec{r} \times \frac{\partial \vec{P}}{\partial t}$$

$$= \int d^3x \left\{ \left[-\vec{r} \times (\rho \vec{E} + \vec{j} \times \vec{B}) \right] + \epsilon^{ijk} r^j \nabla^l \sigma^{lk} \right\}$$

The first term is the opposite of the electromagnetic torque on a charge distribution; this term transfers angular momentum from the ~~fields~~ to charges. Away from charges, we have only the second term

$$\frac{dL}{dt} = \int d^3x \;\; \epsilon^{ijk} r^j \nabla^l \sigma^{lk}$$

Assuming that the distribution of fields is localized, we can integrate by parts

$$= \int d^3x \left\{ -\epsilon^{ijk} (\nabla^l r^j) \, \sigma^{lk} \right\}$$

$$= \int d^3x \left\{ -\epsilon^{ijk} \; \delta^{lj} \; \sigma^{lk} \right\}$$

$$= \int d^3x \left\{ -\epsilon^{ilk} \sigma^{lk} \right\} = 0$$

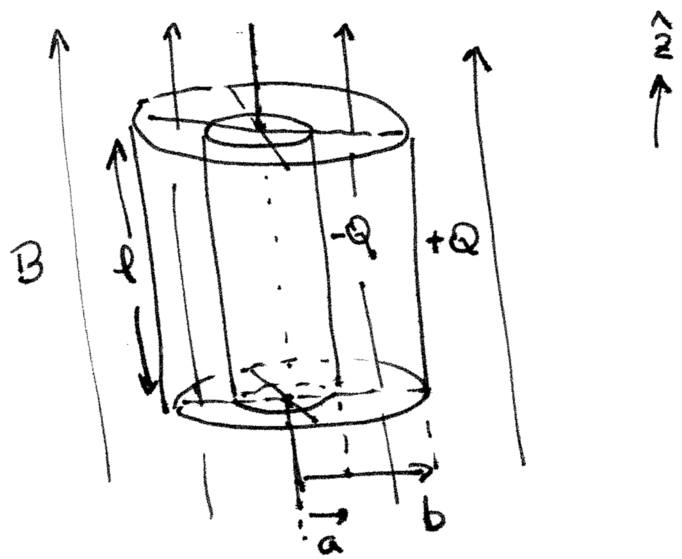
since σ^{lk} is symmetric. Angular momentum is actually locally conserved, but it would take too many indices to prove that here.

Using this expression for $\vec{L}$, I can resolve some unfinished business from the first lecture. I showed you an example with a circular loop of radius a, with a charge $Q = 2\pi\rho a$, mounted in a plane normal to a constant magnetic field $\vec{B} \parallel \hat{z}$. I showed you that, when we turn off the magnetic field, the loop begins to rotate. When the magnetic field is reduced to zero, the loop acquires angular momentum

$$L = \pi a^3 \rho B \hat{z} = \frac{a^2}{2} Q B \hat{z}$$

This angular momentum must have come from the angular momentum in the original electromagnetic field.

Actually, in this case, it is very awkward to do the integrals and prove this.* Griffiths suggests an easier example. Consider a set of two coaxial cylinders with charge $+Q$ and $-Q$, with radii a, b and length ℓ, and $\ell \gg b > a$, immersed in a magnetic field $B\hat{z}$



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* in particular $r \times (E \times B) \sim O(\frac{1}{r})$ as $r \to \infty$, so we must worry about the surface at ∞

If B is changed, an electric field is induced. At radius r,

$$\oint d\vec{l}\cdot\vec{E} = 2\pi r E_\phi = -\pi r^2 \frac{dB}{dt} \Rightarrow E_\phi = -\frac{r}{2}\frac{dB}{dt}$$

The torque on the outer cylinder is

$$\tau_{out} = -b\cdot Q\cdot\left(\frac{b}{2}\frac{dB}{dt}\right)$$ (positive if $\frac{dB}{dt}<0$

The torque on the inner cylinder is

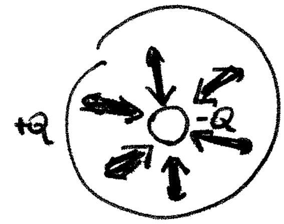
$$\tau_{in} = -a(-Q)\left(\frac{a}{2}\frac{dB}{dt}\right) = +Q\frac{a^2}{2}\frac{dB}{dt}$$

so if the field is turned off the apparatus acquires angular momentum

$$\vec{L} = \frac{1}{2}Q(b^2-a^2)B\hat{z}$$

Now, is this the angular momentum originally in the field. In this configuration, $\vec{E}$ is radial inward between the two cylinders

$$\vec{E} = -\frac{1}{2\pi\epsilon_0}\left(\frac{Q}{\ell}\right)\frac{\hat{r}}{r}$$



so

$$\vec{E}\times\vec{B} = +\hat{\phi}\,\frac{1}{2\pi\epsilon_0}\,\frac{Q}{\ell}\,\frac{1}{r}\cdot B$$

then the angular momentum density

$$\vec{\mathcal{L}} = \epsilon_0\vec{r}\times(\vec{E}\times\vec{B}) = +\hat{z}\,\frac{1}{2\pi}\,\frac{Q}{\ell}\,B$$

we must integrate this over the volume between the cylinders, That is easy, because $\vec{\mathcal{L}}$ is constant. The volume is

$$\int d^3x = \pi(b^2-a^2)\cdot \ell$$

so the original field angular momentum was

$$\vec{L} = \int d^3x\, \vec{\mathcal{L}} = \hat{z}\,\frac{(b^2-a^2)}{\ell}\,Q\,B$$

in precise agreement with the final angular momentum!