Maxwell's Equations

Jan 24

After this long mathematical digression, let's go back to our search for the full time-dependent equations for electric and magnetic fields. We have so far

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \text{electrostatics}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{electrostatics + Faraday}$$

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{J}\end{aligned}\right\} \text{ magnetostatics}$$

Now, there is something wrong with these equations. The equation for $\vec{\nabla}\times\vec{B}$ has an unfortunate property: If we take its divergence

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot\mu_0\vec{J}$$

$$0 = \mu\vec{\nabla}\cdot\vec{J}$$

so the last equation implies that $\nabla\vec{J}=0$, i.e. that $\vec{J}$ is a stationary current. This would be fine for magnetostatics, where our whole formalism assumed that currents were time-independent. However, it is wrong for dynamics.

As we discussed last term, the general equation for electric charge conservation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

The integral form of this expression is

$$\frac{d}{dt} Q_V = - \Phi_S$$

where
$$Q_V = \int_V d^3x \, \rho = \text{charge in } V$$
$$\Phi_S = \int_V d^3x \, \vec{\nabla}\vec{j} = \int_{S=\partial V} d^2x \, \hat{n}\cdot\vec{j} \quad \text{flux of charge out of } V$$

This equation expresses the fact that the total electric charge in any region can change as charge flows from one place to another. A process in which charge moves around requires $\vec{\nabla}\cdot\vec{j} \neq 0$.

Can we write electric and magnetic field equations that are compatible with a general current.
Maxwell suggested replacing the last equation with

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{j} + \mu_0\epsilon_0 \frac{\partial E}{\partial t}$$

Then $$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 \Rightarrow \mu_0 \vec{\nabla}\cdot\left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0$$

$$\vec{\nabla} \cdot \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{E} = \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\epsilon_0} \right)$$

so the constraint we obtain is

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

which is just right. Similarly

$$\nabla \cdot (\vec{\nabla} \times \vec{E}) = 0 = \vec{\nabla} \left(- \frac{\partial \vec{B}}{\partial t} \right) = - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{B} = 0$$

so the other two equations are consistent as they stand.

This gives:

Maxwell's <u>equations</u>:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

Are these reasonable equations, or do we have to modify them still further? Ultimately, experiment is the judge. But I would like to show you, in the rest of this lecture, that these equations pass some quite nontrivial tests that we would require of a fundamental theory of electromagnetism.

An important regularity of Nature is the conservation of energy. We should expect that any fundamental equations for electricity and magnetism should respect a conserved energy. Do these? We can check. If we have a proposal for the form of the energy. I propose that we try

$$E = \int d^3x \left[\tfrac{1}{2}\epsilon_0 (\vec{E})^2 + \frac{1}{2\mu_0}(\vec{B})^2 \right]$$

the sum of electrostatic and magnetostatic energy. Let's consider an array of charges and fields contained in a finite volume, so there is no problem with escape of energy to infinity.

Then compute

$$\frac{dE}{dt} = \int d^3x \left\{ \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} \right\}$$

$$= \int d^3x \left\{ \epsilon_0 \vec{E} \cdot \left\{ \frac{1}{\epsilon_0 \mu_0} \vec{\nabla} \times \vec{B} - \frac{1}{\epsilon_0} \vec{J} \right\} + \frac{1}{\mu_0} \vec{B} \cdot (-\vec{\nabla} \times \vec{E}) \right\}$$

$$= \int d^3x \left\{ \frac{1}{\mu_0} \epsilon^{ijk} [E^i (\nabla^j B^k) + \nabla^j E^i B^k] - \vec{E} \cdot \vec{J} \right\}$$

$$= \int d^3x \left\{ \frac{1}{\mu_0} (\vec{\nabla})^i [-\epsilon^{ijk} E^j B^k] - \vec{E} \cdot \vec{J} \right\}$$

$$= \int d^3x \left\{ -\vec{\nabla} \left(\frac{1}{\mu_0} \vec{E} \times \vec{B} \right) - \vec{E} \cdot \vec{J} \right\}$$

The first term is a surface term at ∞, which can be ignored for a system of localized fields. This leaves

$$\frac{dE}{dt} = - \int d^3x \; \vec{E}\cdot\vec{j}$$

The right-hand side is exactly the work per unit time done by $\vec{E}$ fields on charges. So

away from electric charges $\quad \frac{dE}{dt} = 0$

in the vicinity of charges $\quad \frac{d}{dt}(E + E_{matter}) = 0$

just as we wanted.

A second check on Maxwell's equations is that they respect a conserved momentum. As a candidate for this momentum, I propose

$$\vec{P} = \int d^3x \; \epsilon_0 \, \vec{E}\times\vec{B}$$

notice that the integrand is a cross-product of a vector and a pseudovector. $\vec{E}$ has a physical direction, but the direction of $\vec{B}$ depends on the right-hand rule convention. However, in $\vec{E}\times\vec{B}$, the right-hand rule is applied twice, and so this is a physical vector, as would be required for momentum.

So compute $\quad \frac{d\vec{P}}{dt} = \int d^3x \; \epsilon_0 \left(\frac{\partial\vec{E}}{\partial t}\times B + \vec{E}\times\frac{\partial\vec{B}}{\partial t} \right)$

then

$$\frac{d\vec{P}}{dt} = \int d^3x \quad \epsilon_0 \left\{ \left(\frac{1}{\epsilon_0 \mu_0} \vec{\nabla}\times\vec{B} - \frac{1}{\epsilon_0}\vec{j} \right) \times \vec{B} + \vec{E}\times(-\vec{\nabla}\times\vec{E}) \right\}$$

$$= \vec{F}_j + \vec{F}_B + \vec{F}_E$$

where

$$\vec{F}_j = -\int d^3x \; \vec{j}\times\vec{B}$$

$$\vec{F}_B = -\int d^3x \; \frac{1}{\mu_0} \; \vec{B}\times(\vec{\nabla}\times\vec{B})$$

$$\vec{F}_E = -\int d^3x \; \epsilon_0 \; \vec{E}\times(\vec{\nabla}\times\vec{E})$$

Simplify $\vec{F}_E$:

$$(\vec{F}_E)^i = -\int d^3x \quad \epsilon_0 \left[(\nabla^i \vec{E})\cdot\vec{E} - (\vec{E}\cdot\vec{\nabla}) E^i \right]$$

$$= -\int d^3x \quad \epsilon_0 \left\{ \tfrac{1}{2}\nabla^i E^2 - \nabla^j (E^i E^j) + E^i (\vec{\nabla}\cdot\vec{E}) \right\}$$

the first two terms integrate out to a surface at infinity. The third term is

$$= -\int d^3x \quad \epsilon_0 \; E^i \; \frac{\rho}{\epsilon_0}$$

similarly,
$$\vec{F}_B = -\int d^3x \; \frac{1}{\mu_0} \; \vec{B}(\vec{\nabla}\cdot\vec{B}) = 0$$

so

$$\frac{d\vec{P}}{dt} = -\int d^3x \left\{ \rho\vec{E} + \vec{j}\times\vec{B} \right\}$$

which is exactly equal and opposite to the force on electric charges due to the $\vec{E}$ and $\vec{B}$ fields. Thus,

away from electric charges $\qquad \dfrac{d\vec{P}}{dt} = 0$

in the vicinity of charges $\qquad \dfrac{d}{dt}(\vec{P} + \vec{P}_{matter}) = 0$

So, Maxwell's equations pass the basic test of providing a conserved energy and momentum for the electric and magnetic fields.

I have more to say about the energy and momentum of electromagnetic fields, but first I would like to write the form of Maxwell's equations which is useful in dealing with polarizable media. In the static case, we described polarizable media by separating out the "bound charge" associated with polarization from the "free charge" that is under our direct control.

$$\rho = \rho_f + \rho_b$$

we found ~~that~~ $\rho_b = -\vec{\nabla}\cdot\vec{P}$, and we defined

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Similarly, we separated the "bound current" associated with magnetization

$$\vec{J} = \vec{J}_f + \vec{J}_{b,M}$$

and we found

$$\vec{J}_{b,M} = \vec{\nabla}\times\vec{M}$$

We then defined

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

In the time-dependent case, the above expression for $\vec{J}$ is not quite right. In a situation where $\partial\vec{P}/\partial t \neq 0$

$$\frac{\partial}{\partial t}\rho_b = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P})$$

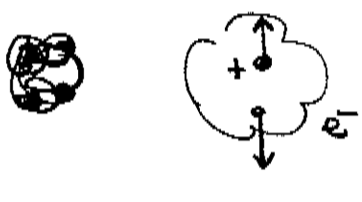
is not zero in general, and so a bound current must flow to represent the movement of bound charge. The expression

$$\vec{J}_{b,P} = \frac{\partial\vec{P}}{\partial t}$$

happily satisfies

$$\frac{\partial}{\partial t}\rho_b + \vec{\nabla}\cdot\vec{J}_{b,P} = 0.$$

$\vec{J}_{b,P}$ is the current associated with atoms increasing their dipole moments:



$$\frac{\partial\vec{P}}{\partial t} = \underbrace{n}_{\text{atoms/cm}^3}\cdot q\cdot\vec{v} = \vec{J}$$

The magnetization current $\vec{J}_{b,M}$ is just fine: $\vec{M}$ produces no net charge density, and $\vec{\nabla}\cdot\vec{J}_{b,M} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) = 0$ even in dynamical situations.

so, decompose

$$\rho = \rho_f + \rho_b \qquad \vec{j} = \vec{j}_f + \vec{j}_{bP} + \vec{j}_{bM}$$

and insert these expressions into Maxwell's equations. The two equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

are of course unchanged. The inhomogeneous equations become

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho_f - \vec{\nabla}\cdot\vec{P}\right)$$

$$\hookrightarrow \quad \vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho_f$$

$$\hookrightarrow \quad \vec{\nabla}\cdot\vec{D} = \rho_f$$

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\left(\vec{j}_f + \frac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}\right)$$

$$\hookrightarrow \quad \vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \frac{\partial}{\partial t}(\epsilon_0\vec{E} + \vec{P}) + \vec{j}_f$$

$$\hookrightarrow \quad \vec{\nabla}\times\vec{H} = \frac{\partial}{\partial t}\vec{D} + \vec{j}_f$$

Maxwell's consistency condition reads:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0 = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) + \vec{\nabla}\cdot\vec{j}_f$$

$$0 = \frac{\partial}{\partial t}\rho_f + \vec{\nabla}\vec{j}_f$$

which is exactly the condition of electric charge conservation in the <u>free</u> charges and currents.

Thus we arrive at Maxwell's equations in media:

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad\qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad\qquad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_f$$

These equations must be supplemented with "constitutive relations" expressing the dependence of D, H on E, B. For linear media:

$$\vec{D} = \epsilon \vec{E} \qquad\qquad \vec{B} = \mu \vec{H}$$

Our estimate of ϵ and μ from last term assumed a static situation or, at least, that the applied fields changed sufficiently slowly that the atoms could easily respond. In dynamic situations, we will often have to worry about whether electric and magnetic fields are changing too rapidly for this approximation to be valid. In many circumstances, the dependence of ε, μ on $\underline{\omega}$ is an important part of the physics. For

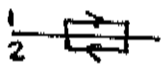
$$\omega \gg (100\,\text{eV})/\hbar = 1.5\times 10^{17}/\text{sec}$$

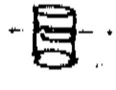
$\varepsilon(\omega), \mu(\omega) \to \varepsilon_0, \mu_0$ in any material.

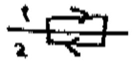
At some engineering schools, you will see people walk around in T-shirts that say, "And God said..." and then giving the equations above. Actually (though I don't presume to speak for the Almighty) it is doubtful that the Creator gave a thought to $\vec{D}$ and $\vec{H}$. Once one has the more fundamental fields $\vec{E}$ and $\vec{B}$ — and quantum mechanics, which governs the properties of electrons and nuclei — the special features of $\vec{D}$ and $\vec{H}$ follow automatically.

Nevertheless, $\vec{D}$ and $\vec{H}$ are useful in solving problems with polarizable media. Last term, we studied problems that involved boundaries between media. For this, it was useful to know the boundary conditions obeyed by the various fields. For the dynamical case, we can evaluate these boundary conditions in the same way, by applying Maxwell's equations to small cylinders or loops embedded in the surface. Assuming no surface <u>free</u> charges or currents, and assuming that $\vec{E}, \vec{B}$ are not divergent at the surface, we have:

$\vec{\nabla}\cdot\vec{D} = \rho_f$  $\rightarrow \quad D_\perp|_1 = D_\perp|_2$

$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$  $\rightarrow \quad \vec{E}_\parallel|_1 = \vec{E}_\parallel|_2$

$\vec{\nabla}\cdot\vec{B} = 0$  $\rightarrow \quad B_\perp|_1 = B_\perp|_2$

$\vec{\nabla}\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t} + \vec{J}_f$  $\rightarrow \quad \vec{H}_\parallel|_1 = \vec{H}_\parallel|_2$

just as in the static case.