Jan 19

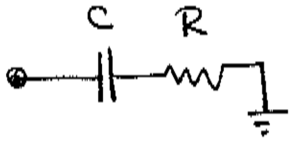
Green's Functions

Now that we have understood how to compute a particular solution to a circuit equation, let's now return to the problem of computing the transients. A typical problem we are interested in is: We have a circuit completely at rest: $Q=0$, $I=0$. We apply a time-dependent signal $V(t)$ beginning at $t=0$. What happens?

Mathematically, this problem has the form

$$V(t) = \text{differential operator applied to } I \text{ or } Q$$

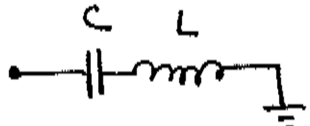
For example, for the simple circuit



$$V(t) = R\frac{dQ}{dt} + \frac{1}{C}Q$$

In general, write $\quad V(t) = \mathcal{D}\cdot Q$

For the circuit



$$\mathcal{D}Q = L\frac{d^2Q}{dt^2} + \frac{1}{C}Q$$

To solve this problem, use the following observation. Let's say that we can find the solution to the equation

$$\mathcal{D} Q = \delta(t-t_0)$$

the response to a δ-function signal applied at $t=t_0$

Call this solution the "Green's function"

$$G(t;t_0)$$

Then the solution to our original equation is

$$Q(t) = \int dt_0 \; G(t;t_0) V(t_0)$$

The proof is simple

$$\mathcal{D} Q = \int dt_0 \; \mathcal{D} G(t;t_0) \; V(t_0)$$

$$= \int dt_0 \; \delta(t-t_0) \; V(t_0) = V(t)$$

And, there is a bonus to writing the solution in this way. We can insure that V(t) obeys the correct initial conditions by choosing $G(t;t_0)$ to obey correct initial conditions!

Let me illustrate this with the first circuit on p. 1.

We wish to solve
$$V(t) = R \frac{dQ}{dt} + \frac{1}{C} Q$$

with the initial condition $Q=0$ at $t=0$. Since this is a first-order differential equation, there is a unique solution with this initial condition. To find it, study the equation for the Green's function

$$R \frac{d}{dt} G(t;t_0) + \frac{1}{C} G(t;t_0) = \delta(t-t_0)$$

The general solution to the homogeneous equation is

$$G = A e^{-t/RC}$$

This is the form of the solution away from $t = t_0$. Just at $t = t_0$, we can integrate over the $\delta()$ function

$$\int_{t_0-\epsilon}^{t_0+\epsilon} dt \left\{ R \frac{d}{dt} G + \frac{1}{C} G \right\} = \int_{t_0-\epsilon}^{t_0+\epsilon} dt \; \delta(t-t_0)$$

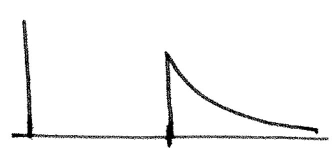
$$= R\left[G(t_0+\epsilon;t_0) - G(t_0-\epsilon;t_0)\right] + O(\epsilon) = 1$$

so

$G(t;t_0)$ has a <u>step</u> at t_0 with $\Delta G = \frac{1}{R}$

If we now require $G(t;t_0) = 0 \quad t < t_0$, this gives the unique solution

$$G(t;t_0) = \begin{cases} 0 & t < t_0 \\ \frac{1}{R} e^{-(t-t_0)/RC} & t > t_0 \end{cases}$$



then

$$Q(t) = \int_{t>t_0} dt_0 \;\; \frac{1}{R} e^{-(t-t_0)/RC} \;\; V(t_0)$$

$$Q(t) = \int_{\infty}^{t} dt_0 \;\; \frac{1}{R} e^{-(t-t_0)/RC} \; V(t_0)$$

As a check, we can compute the response to a constant voltage V_0 applied starting at $t=0$

$$Q(t) = \int_0^t dt_0 \;\; \frac{1}{R} \;\; e^{-(t-t_0)/RC} \;\; V_0$$

$$= \frac{1}{R} \, e^{-t/RC} \, V_0 \int_0^t dt_0 \;\; e^{t_0/RC}$$

$$= \frac{V_0}{R} \, e^{-t/RC} \; (e^{t/RC} - 1) \cdot RC$$

$$Q(t) = CV_0 \; (1 - e^{-t/RC})$$

which is the result we are familiar with.

A Green's function which satisfies

$$G(t;t_0) = 0 \quad \text{for} \quad t < t_0$$

is called a "retarded Green's function". There is a commonsense principle which in physics is called causality: A system responds after we act on it, not before. The retarded

Green's function implements causality mathematically in the response of a linear system. If $G_R(t;t_0)$ is a retarded Green's function

$$Q(t) = \int dt_0 \; G_R(t;t_0)\, V(t_0)$$

$$= \int_{-\infty}^{t} dt_0 \; G_R(t;t_0)\, V(t_0)$$

so $Q(t)$ responds only to signals applied at $t_0 < t$.

Sometimes the condition that $Q(t)$, $\frac{d}{dt}Q(t), \ldots = 0$ before a signal is applied is called the "retarded boundary condition". Its unphysical opposite, that Q, $\frac{d}{dt}Q, \ldots = 0$ after the signal is applied, is called the "advanced boundary condition." (In quantum field theory, it is sometimes necessary to account for antiparticles by using a mixture of these conditions, called the "Feynman boundary condition".)

Let's look at another example of this type, the undamped LC circuit:

$$V(t) = L \frac{d^2Q}{dt^2} + \frac{1}{C} Q$$

The equation for the Green's function is

$$L \frac{d^2}{dt^2} G(t;t_0) + \frac{1}{C} G(t,t_0) = \delta(t-t_0)$$

The general solution of this equation, for $t \neq t_0$, is

$$G = A \sin \omega_0 t + B \cos \omega_0 t$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Integrating through the δ-function, we find

$$L \left[\frac{d}{dt} G(t_0+\varepsilon) - \frac{d}{dt} G(t_0-\varepsilon) \right] = 1$$

or
$$\Delta\left(\frac{dG}{dt}\right) = \frac{1}{L} \qquad \text{and} \qquad \Delta G = 0$$

$\Delta G = 0$ because a discontinuity in dG/dt does not immediately integrate up to a discontinuity in G. The <u>retarded</u> <u>Green's</u> function, with $Q = 0$, $dQ/dt = 0$ for $t < t_0$, is then

$$G_R(t;t_0) = \begin{cases} 0 & t < t_0 \\ A \sin \omega_0 (t-t_0) & t > t_0 \end{cases}$$

with A set by the condition on $\Delta(dG/dt)$:

$$\frac{1}{L} = A\omega_0 \quad \rightarrow \quad A = \sqrt{\frac{C}{L}}$$

again

$$G_R(t;t_0) = \sqrt{\frac{C}{L}} \sin \omega_0 (t-t_0) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

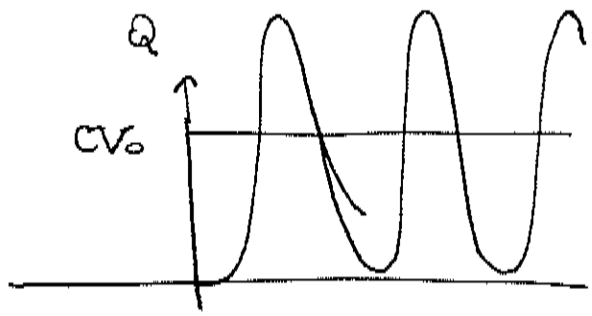
Again, check this by computing the response to a step in voltage:

$$V(t) = \begin{cases} 0 & t<0 \\ V_0 & t>0 \end{cases}$$

$$Q(t) = \int_0^t dt_0 \sqrt{\frac{C}{L}} \sin\omega_0(t-t_0) \cdot V_0 \qquad t>0$$

$$= V_0 \sqrt{\frac{C}{L}} \frac{1}{\omega_0} \cos\omega_0(t-t_0) \Big|_{t_0=0}^{t_0=t}$$

$$Q(t) = CV_0 (1 - \cos\omega_0 t) \qquad t \geq 0$$



This is, correctly, an undamped oscillation about the equilibrium value $Q = CV_0$ with frequency $\omega_0 = \frac{1}{\sqrt{LC}}$. Adding a small resistor R, we would find

In the two problems we have just discussed, the differential equation has no preferred origin of time, and thus

$$G_R(t;t_0) = G_R(t-t_0)$$

Then

$$Q(t) = \int dt_0 \; G(t-t_0)\, V(t_0)$$

which is a convolution. So there should be a relation between the method of Green's functions and the Fourier transform method discussed in the previous lecture. Can we compute a Green's function by Fourier transformation?

To discuss this topic, we should review one more result from mathematics, another topic from the theory of complex variables. It may seem abstruse at first, but it leads to one of the real magic tricks of mathematical physics.

Let $f(z)$ be an analytic function, ~~that~~ is, a function of the two variables x, y which depends only on the combination

$$z = x + iy$$

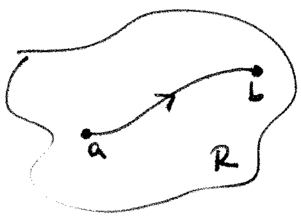
and not on $\bar{z} = x - iy$. We encountered these functions in our study of the solution of Laplace's equation by conformal mapping.

An analytic function obeys the Cauchy-Riemann equations

$$\frac{\partial}{\partial x}\,\mathrm{Re}\,f - \frac{\partial}{\partial y}\,\mathrm{Im}\,f = 0$$

$$\frac{\partial}{\partial x}\,\mathrm{Im}\,f + \frac{\partial}{\partial y}\,\mathrm{Re}\,f = 0$$

Let $f(z)$ be nonsingular in a region R. Let C be a contour in this region



and consider $\displaystyle\int_a^b dz\; f(z)$

Taking the real and imaginary part,

$\mathrm{Re}\left[\int_a^b dz\, f(z)\right] = \int dx\, \mathrm{Re} f$ [redacted] $dy\, \mathrm{Im} f = \int d\vec{l}\cdot \vec{F}_R$

$$\mathrm{Im}\left[\int_a^b dz\, f(z)\right] = \int dx\, \mathrm{Im} f + dy\, \mathrm{Re} f = \int d\vec{l}\cdot \vec{F}_I$$

where $\vec{F}_R, \vec{F}_I$ are the vector fields

$$\vec{F}_R = (\mathrm{Re} f, -\mathrm{Im} f) \qquad \vec{F}_I = (\mathrm{Im} f, \mathrm{Re} f)$$

now $\vec{\nabla}\times\vec{F}_R = \left(-\frac{\partial}{\partial x}\mathrm{Im} f\right.$ [redacted] $\left.\frac{\partial}{\partial y}\mathrm{Re} f\right)\hat{z} = 0$

$\vec{\nabla}\times\vec{F}_I = \left(\frac{\partial}{\partial x}\mathrm{Re} f\right.$ [redacted] $\left.\frac{\partial}{\partial y}\mathrm{Im} f\right)\hat{z} = 0$

by the Cauchy-Riemann eqs.

so $\oint_C d\vec{l}\cdot\vec{F}_R = 0 \qquad \oint_C d\vec{l}\cdot\vec{F}_I = 0 \qquad$ or $\qquad \oint_C dz\, f(z) = 0$

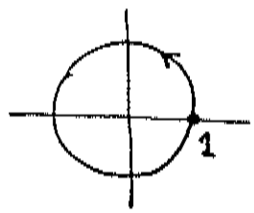
for any closed contour in R. This also implies that the value of

$$\int_a^b dz\, f(z)$$

does not depend on the path chosen, as long as it does not cross and singularities of f(z). Here is a simple example: Let

$f(z) = az^n \quad (n \geq 0) \qquad C =$ unit circle

$z = e^{i\phi}$ on this path



$$\oint dz\; az^n = \int_0^{2\pi} i\,d\phi\; a\, e^{i(n+1)\phi}$$

$$= 0 \quad \text{if } n \geq 0$$

However, for the singular function $f(z) = \frac{a}{z}$

$$\oint dz\; \frac{a}{z} = \int_0^{2\pi} i\,d\phi \cdot a \cdot 1 = 2\pi i$$

In fact, if f(z) has a "Laurent series" expansion

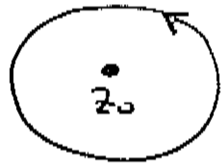
$$f(z) = \frac{a_{-n}}{z^n} + \frac{a_{-(n-1)}}{z^{n-1}} + \dots + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$$

then

$$\oint dz\, f(z) = 2\pi i\, a_{-1}$$

A singularity of $f(z)$ of the form $\frac{A}{z-z_0}$ is called a "simple pole" of $f(z)$; similarly, a singularity $\frac{B}{(z-z_0)^2}$ is a "double pole", etc.

In general, if $f(z)$ has a pole-type singularity at $z = z_0$ and we integrate around the singularity



$$\oint dz \; f(z) = 2\pi i \, A$$

where A is the coefficient of the single pole term $\frac{A}{z-z_0}$.

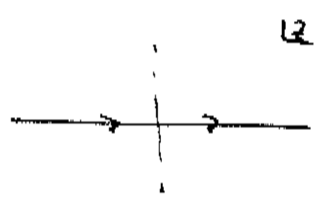
A is called the "residue of $f(z)$ at z_0".

We can use this trick to do all kinds of challenging integrals. For example, consider

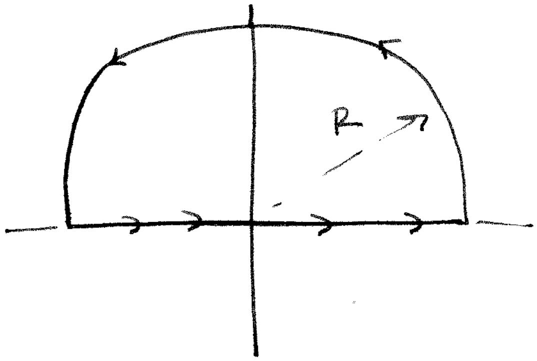
$$I = \int_{-\infty}^{\infty} dx \; \frac{1}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \Big|_{-\infty}^{\infty} = \frac{\pi}{a}$$

We can alternatively do this integral by noticing that I is the complex integral

$$I = \int dz \; \frac{1}{z^2+a^2}$$



taken along the real axis in the complex plane. Add a return path at a large radius R:



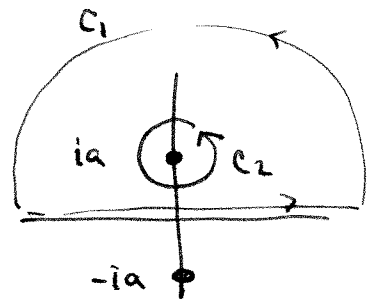
The integral along this new path is $\sim \; R\frac{1}{R^2} \to 0$ as $R \to \infty$, so we can add this path without affecting the value of I. But the new path is a closed contour, so we can contract it onto the singularities of $f(z)$. In this case

$$f(z) = \frac{1}{z^2+a^2} = \frac{1}{(z+ia)(z-ia)}$$

so $f(z)$ has <u>poles</u> at $z = \pm ia$. Near $z = +ia$

$$f(z) \sim \frac{1}{(z-ia)} \cdot \frac{1}{2ia} + (\text{regular})$$

so



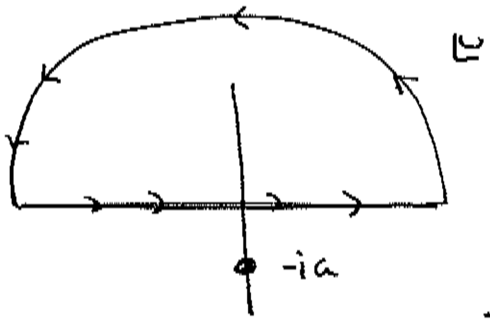
$$I = \oint_{C_1} dz\, f(z) = \oint_{C_2} dz\, f(z)$$

$$= 2\pi i \cdot \frac{1}{2ia} = \frac{\pi}{a} \quad !$$

In a course in complex variable theory, you would see many more complex examples. However, most interesting to us is the <u>discontinuous</u> <u>integral</u> of Cauchy!

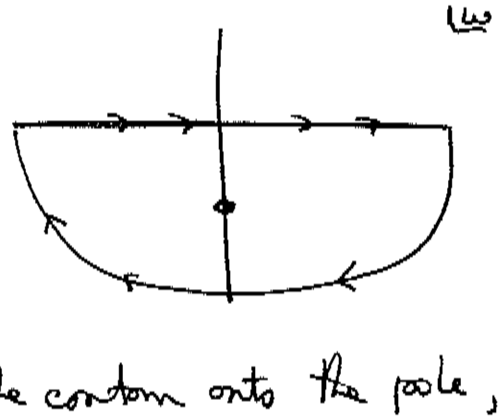
$$I = \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega + ia} = \begin{cases} 0 & t < 0 \\ -2\pi i e^{-at} & t > 0 \end{cases}$$

Here is the derivation: The function $f(\omega)$ has a pole at $\omega = -ia$. For $t < 0$ we can make the integral an integral over a closed contour by adding a loop in the upper half plane:



on the loop $\omega = R\cos\theta + iR\sin\theta$, so $|e^{-i\omega t}| \sim e^{R\sin\theta\, t}$ which $\to 0$ for $R \to \infty$, $t < 0$. This contour encloses no poles, so $I = 0$ for $t < 0$.

For $t > 0$, this added contour is not small. Instead, we must close the contour by adding the loop



$$|e^{-i\omega t}| \sim e^{-R\sin\theta\, t}$$

Then, contracting the contour onto the pole, we find

$$I = \underset{\substack{\uparrow \\ \text{wrong way} \\ \text{around}}}{-2\pi i} \cdot e^{-i\omega t}\Big|_{\omega = -ia} = -2\pi i\, e^{-at}$$

The amazing result is that the integral has discontinuous behavior and is zero for $t<0$.

Now, what does this have to do with our circuit problem? Let's try to find the Green's function for the RC circuit on p.1 by using the Fourier transform. As on p.3, we need to solve

$$R\frac{d}{dt}G_R(t;t_0) + \frac{1}{C}G_R(t;t_0) = \delta(t-t_0)$$

subject to the condition $G_R(t;t_0) = 0$ for $t<t_0$. Introduce the Fourier representation

$$G_R(t;t_0) = \int\frac{d\omega}{2\pi}\,\tilde{G}_R(\omega)\,e^{-i\omega t}$$

$$\frac{d}{dt}G_R(t;t_0) = \int\frac{d\omega}{2\pi}\,-i\omega\,\tilde{G}_R(\omega)\,e^{-i\omega t}$$

$$\delta(t-t_0) = \int\frac{d\omega}{2\pi}\,e^{-i\omega(t-t_0)}$$

so the above equation becomes:

$$\int\frac{d\omega}{2\pi}\,e^{-i\omega t}\left\{(-i\omega R + \tfrac{1}{C})\,\tilde{G}_R(\omega) = e^{i\omega t_0}\right\}$$

and we can solve for $\tilde{G}_R$ very simply

$$\tilde{G}_R(\omega) = \frac{e^{i\omega t_0}}{-i\omega R + \frac{1}{C}} = \left(\frac{i}{\omega}\right)\left(\frac{e^{i\omega t_0}}{R + i/\omega C}\right)$$

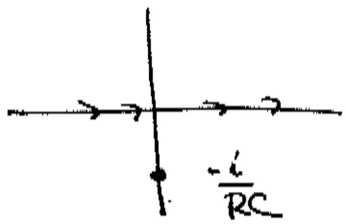
Then

$$G_R(t;t_0) = \int \frac{d\omega}{2\pi} \frac{1}{(-i\omega R + \frac{1}{C})} e^{-i\omega(t-t_0)}$$

This is exactly of the form of Cauchy's discontinuous integral, with the singularity of the integrand at

$$\omega = -i\frac{1}{RC}$$

in the lower half of the complex plane. So, if we take the integral to run along the real axis



$$G_R(t;t_0) = \begin{cases} 0 & t < t_0 \\ -2\pi i \cdot \text{Residue}\left(\tilde{G}(\omega),\ \omega = \frac{-i}{RC}\right) & t > t_0 \end{cases}$$

Now, near $\omega = \frac{-i}{RC}$

$$\tilde{G}_R(\omega) \sim \frac{i}{R} \frac{1}{(\omega + \frac{i}{RC})}$$

so for $t > t_0$

$$G_R(t;t_0) = \frac{1}{R} e^{-(t-t_0)/RC}$$

exactly as we found on p. 3.

Let's now return to the solution of the more general equation

$$R \frac{d}{dt} Q + \frac{1}{C} Q = V(t)$$

The solution by Fourier transformation is

$$Q(t) = \int \frac{d\omega}{2\pi} \frac{1}{-i\omega R + \frac{1}{C}} \tilde{V}(\omega) e^{-i\omega t}$$

$$= \int \frac{d\omega}{2\pi} \tilde{G}_R(\omega) \tilde{V}(\omega) e^{-i\omega t}$$

we know that, in real time, this is a convolution

$$Q(t) = \int dt' \; G_R(t-t') V(t')$$

If G_R is taken to be the retarded solution of its differential equation, this expression for $Q(t)$ obeys the causal boundary condition

$$V(t) = 0 \text{ for } t<0 \Rightarrow Q(t) = 0 \text{ for } t<0 \quad !$$

Now do the same exercise for the LC circuit discussed on p. 1 and p. 5. The equation for the Green's function is

$$L \frac{d^2}{dt^2} G_R(t;t_0) + \frac{1}{C} G_R(t;t_0) = \delta(t-t_0)$$

We can be a little smarter and realize that

$$G_R(t;t_0) = G_R(t-t_0)$$

so

$$G_R = \int \frac{d\omega}{2\pi}\, \tilde{G}_R(\omega)\, e^{-i\omega(t-t_0)}$$

then

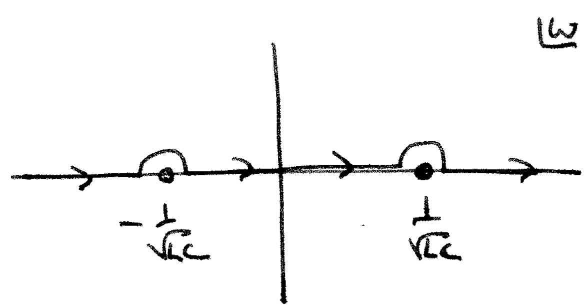
$$\int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t_0)} \left\{ \left((-i\omega)^2 L + \frac{1}{C}\right)\tilde{G}_R = 1 \right\}$$

$$\tilde{G}_R(\omega) = \frac{1}{-\omega^2 L + \frac{1}{C}}$$

and

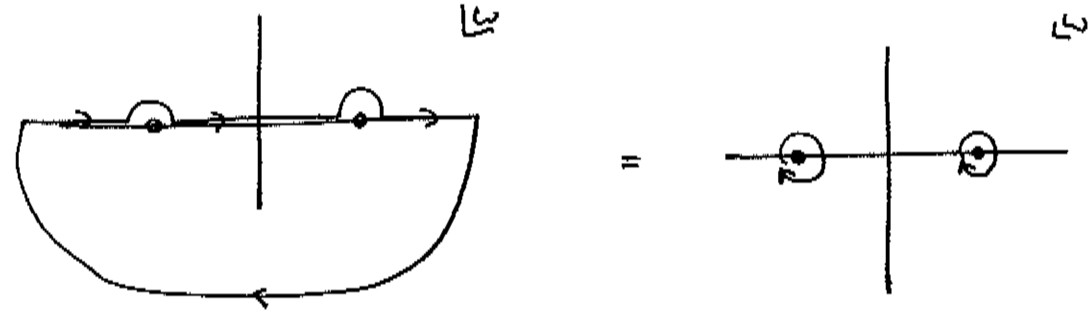
$$G_R(t-t_0) = \int \frac{d\omega}{2\pi} \left(\frac{-1}{\omega^2 L - \frac{1}{C}}\right) e^{-i\omega(t-t_0)}$$

Now, the integrand has singularities at $\omega^2 = \frac{1}{LC}$, $\omega = \pm\frac{1}{\sqrt{LC}}$. These points are on the real axis. So, on what contour should we do the integral. I propose using the integration contour <u>above</u> the singularities:



For $t<t_0$, we would close the contour in the upper half plane and obtain $G_R(t-t_0) = 0$ for $t<t_0$.

For $t > t_0$, we would close the contour in the lower half plane



since
$$\hat{G}_R(\omega) = \frac{-1}{\omega^2 L - \frac{1}{C}} = \frac{-1/L}{\left(\omega - \frac{1}{\sqrt{LC}}\right)\left(\omega + \frac{1}{\sqrt{LC}}\right)}$$

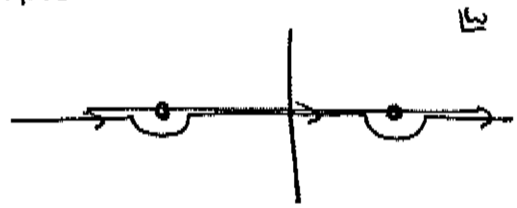
near $\omega = \frac{1}{\sqrt{LC}}$
$$\hat{G}_R(\omega) \sim \frac{1}{\omega - \frac{1}{\sqrt{LC}}} \left(-\frac{1}{L} \cdot \frac{1}{2\frac{1}{\sqrt{LC}}} \right)$$
$$\sim \frac{1}{\omega - \frac{1}{\sqrt{LC}}} \cdot \left(-\frac{1}{2}\sqrt{\frac{C}{L}} \right)$$

near $\omega = -\frac{1}{\sqrt{LC}}$
$$\hat{G}_R(\omega) \sim \frac{1}{\omega + \frac{1}{\sqrt{LC}}} \left(+\frac{1}{2}\sqrt{\frac{C}{L}} \right)$$

so
$$G_R(t-t_0) = \left(-\frac{2\pi i}{2\pi}\right)\left(-\frac{1}{2}\sqrt{\frac{C}{L}}\right) e^{-i(t-t_0)\frac{1}{\sqrt{LC}}}$$
$$+ \left(-\frac{2\pi i}{2\pi}\right)\left(\frac{1}{2}\sqrt{\frac{C}{L}}\right) e^{+i(t-t_0)\frac{1}{\sqrt{LC}}}$$
$$= \frac{1}{2i}\left(e^{i(t-t_0)\frac{1}{\sqrt{LC}}} - e^{-i(t-t_0)\frac{1}{\sqrt{LC}}}\right)\sqrt{\frac{C}{L}}$$
$$= \sqrt{\frac{C}{L}} \sin\left(\frac{(t-t_0)}{\sqrt{LC}}\right)$$

just as on p. 6!

The choice of contour:



gives the advanced Green's function $G_A(t-t_0)$, which (unphysically) satisfies $G_A(t-t_0) = 0$ for $t > t_0$.

This LC circuit is undamped, but a realistic LC circuit will have some damping by resistance. If we add a resistor to the circuit, the Green's function satisfies

$$L\frac{d^2G_R}{dt^2} + R\frac{dG_R}{dt} + \frac{1}{C}G_R = \delta(t-t_0)$$

which is solved by

$$\tilde{G}_R(\omega) = \frac{1}{-\omega^2 L - i\omega R + \frac{1}{C}}$$

The singularities of the function are located at ω st.

$$\omega^2 L + i\omega R - \frac{1}{C} = 0$$

$$\omega = -\frac{i}{2}\frac{R}{L} \pm \left[\frac{1}{LC} - \frac{1}{4}\frac{R^2}{L^2}\right]^{\frac{1}{2}}$$

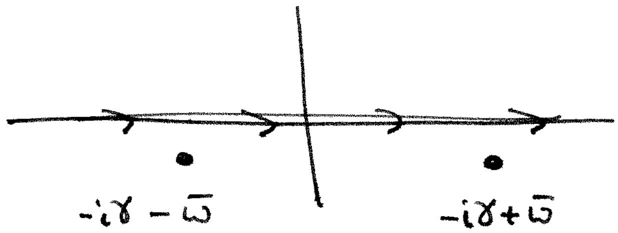
$$= -i\gamma \pm \bar{\omega}$$

in our earlier notation. These poles are in the lower half plane.

The retarded Green's function is given by

$$G_R(t-t_0) = \int \frac{d\omega}{2\pi} \frac{-1/L}{(\omega + i\gamma - \bar{\omega})(\omega + i\gamma + \bar{\omega})} e^{-i\omega(t-t_0)}$$

evaluated on the contour:



Evaluating using the method of residues, we find

$$G_R(t-t_0) = 0 \qquad t < t_0$$

and for $t > t_0$

$$G_R(t-t_0) = \left(\frac{-2\pi i}{2\pi}\right)\left(-\frac{1}{L}\right) \frac{1}{2(\bar{\omega}+i\gamma)} e^{-\gamma(t-t_0)} e^{-i\bar{\omega}(t-t_0)}$$

$$+ \left(\frac{-2\pi i}{2\pi}\right)\left(-\frac{1}{L}\right) \frac{1}{-2(\bar{\omega}+i\gamma)} e^{-\gamma(t-t_0)} e^{-i\bar{\omega}(t-t_0)}$$

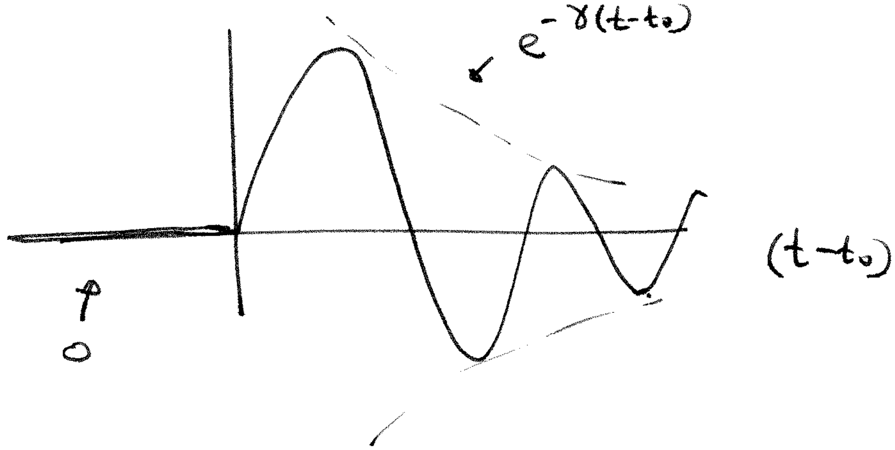
Using $$\frac{1}{\bar{\omega}+i\gamma} = \frac{1}{\bar{\omega}^2+\gamma^2}(\bar{\omega}-i\gamma) = \left(\frac{1}{\sqrt{LC}}\right)^{-1} e^{-i\phi}$$

where $\tan\phi = \dfrac{\gamma}{\bar{\omega}}$

$$G_R(t-t_0) = \sqrt{\frac{C}{L}}\, e^{-\gamma(t-t_0)} \sin\left(\bar{\omega}(t-t_0) + \phi\right)$$

for $t > t_0$

With even a small damping added, it is obvious that the contour of ~~integration~~ should go above the singularities. Then $G_R(t-t_0)$ has the form:



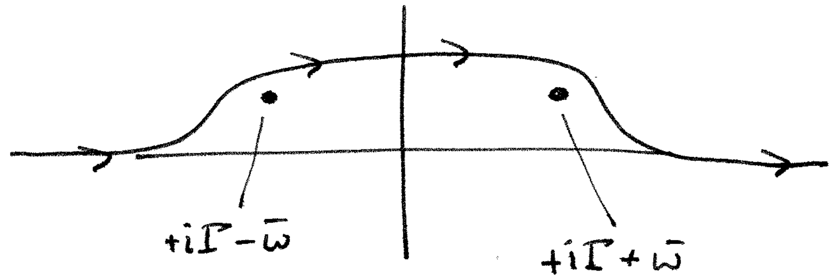
and, again, the solution for a general $V(t)$ that starts from $t=0$ is.

$$Q(t) = \int_0^t dt' \; G_R(t-t')\, V(t')$$

What if the poles on p. 20 were in the upper ξ-plane. Physically, this would correspond to a <u>negative</u> resistance — an instability. What does the mathematics say? Causality requires us to place the contour <u>above</u> the singularities, so that we guarantee

$$G_R(t;t_0) = 0 \qquad t < t_0$$

Thus,



For $t > t_0$, we evaluate G_R by closing the contour in the lower half-plane. This gives

residue $\sim e^{+\Gamma(t-t_0)} e^{+i\omega(t-t_0)}$ residue $\sim e^{+\Gamma(t-t_0)} e^{-i\omega(t-t_0)}$

so the mathematics confirms that there is an exponentially growing instability in this case.

In more general linear systems, we can diagnose instabilities in this way: In Fourier transform space

singularities in the lower ½ plane $\longrightarrow$ stable modes
singularities in the upper ½ plane $\longrightarrow$ <u>un</u>stable modes