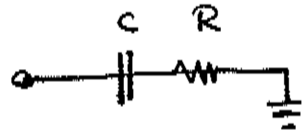
The Fourier Transform

Jan 17

In the previous lecture, we analyzed circuits driven by AC voltage oscillating at a fixed frequency. For example: if



is driven by $V(t) = V_0 \cos \omega t = \mathrm{Re}\, V_0 e^{-i\omega t}$

the current that flows in response is

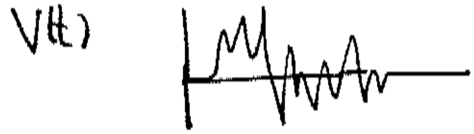
$$I(t) = \mathrm{Re}\, I_0 e^{-i\omega t} \quad \text{with} \quad I_0 = \frac{V_0}{R + i/\omega C}$$

in general

effective impedance $Z(\omega)$

$$I_0 = \frac{V_0}{Z(\omega)}$$

Now, what happens if we drive the circuit with a signal which is a more arbitrary function of time?



There are two aspects to the problem. First, if the behavior of the circuit is described by a differential equation, we might with the find a solution of the differential equation with source term $V(t)$. In the case of the simple circuit above

$$V(t) = R\frac{dQ}{dt} + \frac{1}{C}Q$$

The most general solution to this differential equation is

$$Q(t) = Q_V(t) + Q_0(t)$$

where $Q_V(t)$ is a particular solution to the inhomogeneous equation and $Q_0(t)$ is a general solution to the equation with $V(t) = 0$. For this equation,

$$Q_0(t) = a\, e^{-t/RC}$$

where a is a constant, so the second term falls off in a natural damping time for the circuit. This term is called a "transient". Let's first discuss how to compute the particular solution $Q_V(t)$, then come back and solve for the transient behavior.

Since we know how to find $Q_V(t)$ when $V(t)$ is an oscillation with a definite frequency, let's try to reduce the solution for a general $V(t)$ to this case. If $V(t)$ is periodic with period T, we can represent $V(t)$ by a Fourier series

$$V(t) = \sum_{n=-\infty}^{\infty} V_n\, e^{-i2\pi n t/T}$$

The coefficients in this series are calculated by the formula

$$V_n = \frac{1}{T}\int_0^T dt \; V(t) \; e^{i\,2\pi n t/T}$$

There is a nice consistency check:

$$V_n \overset{?}{=} \frac{1}{T}\int_0^T dt \; e^{i\cdot 2\pi n t/T} \sum_{m=-\infty}^{\infty} V_m \, e^{-i 2\pi m t/T}$$

By the orthogonality of sins and cos's, the integral vanishes unless $n=m$; in that case, the integral is trivial. So

$$\frac{1}{T}\int_0^T dt \; e^{i\cdot 2\pi n t/T} \, e^{-i\,2\pi m t/T} = \delta_{nm}$$

and the check succeeds. In the other direction, we should have

$$V(t) = \sum_{n=-\infty}^{\infty} e^{-2\pi i n t/T} \frac{1}{T}\int_0^T dt' \; e^{2\pi i n t'/T} \, V(t')$$

$$= \int_0^T dt' \left\{ \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-2\pi i n t/T} \, e^{2\pi i n t'/T} \right\} V(t')$$

which is consistent if

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-2\pi i n t/T} \, e^{+2\pi i n t'/T} = \delta(t-t')$$

All of these formulas apply if $V(t)$ is complex. If $V(t)$ is real, the Fourier series coefficients are restricted by

$$V_{-n} = (V_n)^*$$

What if the signal $V(t)$ is not periodic? We can always view it as being periodic with a very large period. So let's try to take the limit $T \to \infty$ of the formulae above and see if we get a sensible result.

Let

$$\omega = \frac{2\pi n}{T} \qquad \text{so} \qquad e^{-2\pi i n t/T} \to e^{-i\omega t}$$

for finite T, the ω are <u>discrete</u> multiples of $\frac{2\pi}{T}$. As $T \to \infty$, these frequencies become continuous. Then

$$\sum_{n=-\infty}^{\infty} \cong \int_{-\infty}^{\infty} dn \longrightarrow \frac{T}{2\pi}\int_{-\infty}^{\infty} d\omega$$

so

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}$$

Let

$$\tilde{V}(\omega) = T\, V_n$$

Then the formulae above go over to

$$V(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \tilde{V}(\omega)\, e^{-i\omega t}$$

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} dt\, V(t)\, e^{i\omega t}$$

These formulae are consistent if

$$V(t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \int dt'\, e^{i\omega t'}\, V(t')$$

i.e. if

$$\int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} = \delta(t-t')$$

and if

$$\tilde V(\omega) = \int dt \; e^{i\omega t} \int \frac{d\omega'}{2\pi}\, e^{-i\omega' t}\, \tilde V(\omega) \, .$$

i.e. if

$$\int dt \; e^{i(\omega-\omega')t} = 2\pi\, \delta(\omega-\omega')$$

The first of these formulae is the $T \to \infty$ limit of the formula at the bottom of p.3. The $T \to \infty$ limit of the formula in the middle of p.3

$$\int_0^T dt \; e^{i\frac{2\pi n}{T}t}\, e^{-i\frac{2\pi m}{T}t} = T\,\delta_{nm} = 2\pi\, \frac{T}{2\pi}\,\delta_{nm}$$

indeed gives the second formula above when we realize that $2\pi/T = \Delta\omega$, the spacing of the discrete ω_n. Then

$$\frac{1}{\Delta\omega}\,\delta_{nm} \to \delta(\omega-\omega')$$

The function $\tilde V(\omega)$ represents the decomposition of the function $V(t)$ into components of definite frequency. $\tilde V(\omega)$ is called the <u>Fourier transform</u> of $V(t)$. Given the simple inversion formulae, $V(t)$ and $\tilde V(\omega)$ are two equivalent representations of the same information, $V(t)$ in time and $\tilde V(\omega)$ in frequency space. If $V(t)$ is real-valued, $\tilde V(\omega)$ obeys the "reality condition" $\quad \tilde{V}(-\omega) = [\tilde{V}(\omega)]^*$.

Notice that the inversion formulas and also the integrals which give δ-functions, contain factors of 2π asymmetrically. Some people like to move around factors of $\sqrt{2\pi}$ to make these formulae symmetrical. In my experience, you will drop many fewer factors of 2π if you just make it a habit to write the 2π with ω:

$$\int \frac{d\omega}{2\pi} \qquad\qquad 2\pi\,\delta(\omega-\omega')$$

Given the Fourier transform of a signal $V(t)$, we can find a solution for the current in any circuit. The response to

$V(t) = \mathrm{Re}\,\tilde{V}(\omega)\,e^{-i\omega t}$ is $\quad I(t) = \mathrm{Re}\,\dfrac{\tilde{V}(\omega)}{Z(\omega)}\,e^{-i\omega t}$

Then, by superposition, a general $V(t)$ yields

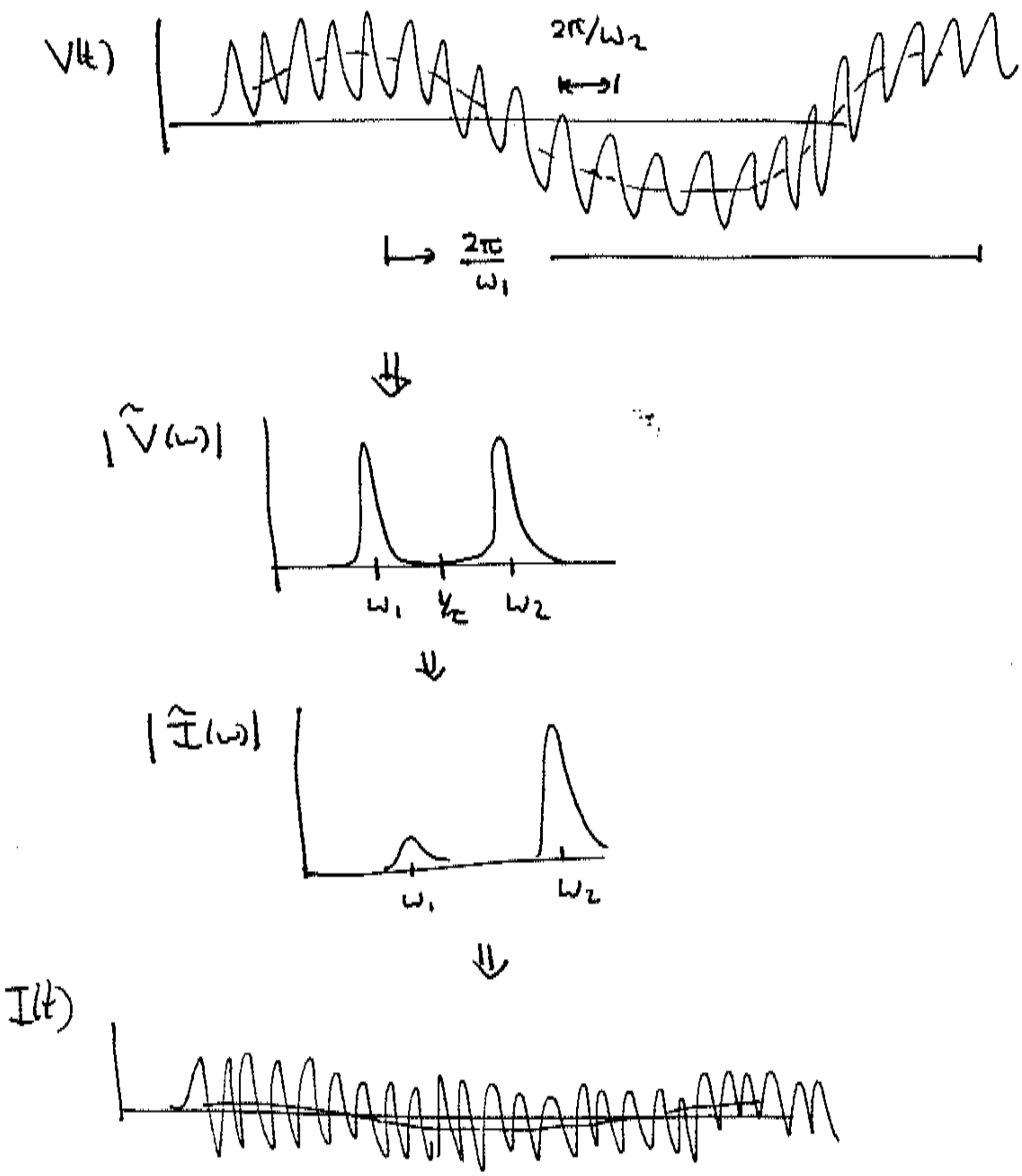
$$I(t) = \int \frac{d\omega}{2\pi}\,\frac{\tilde{V}(\omega)}{Z(\omega)}\,e^{-i\omega t}$$

When we looked at particular circuits, we saw that they could have a different response for different frequencies. For example, the simple circuit on p. 1 has

$$\frac{1}{Z(\omega)} = \frac{1}{R + i/\omega C} = \frac{\omega C}{\omega C R + i}$$

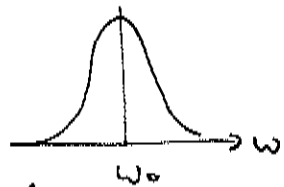
$$\left|\frac{1}{Z(\omega)}\right| = \frac{\omega C}{[(\omega C R)^2 + 1]^{\frac{1}{2}}}$$

For $\omega \gg 1/\tau$ $\tau = RC$, $\frac{1}{Z} \cong \frac{1}{R}$ and the circuit feels no effect of the capacitor. For $\omega \ll 1/\tau$, the capacitor essentially stops the flow of current and so the response is very small. Essentially, this $Z(\omega)$ allows the high frequency components of the signal to pass while damping out the low frequencies. Here is a graphical illustration of this process



It is very useful to learn to think in frequency space, to visualize the decomposition of a given waveform in terms of frequencies and the waveform that results when this frequency spectrum is modified in some way. A useful tool to help this process of visualization is the "Gaussian wavepacket."

$$\tilde{f}(\omega) = e^{-a^2(\omega-\omega_0)^2/2}$$



This is a concentrated set of frequencies, centered on ω_0, with width $\Delta\omega \simeq 1/a$. Going back to "real space" by inverting the Fourier transform, we find

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-a^2(\omega-\omega_0)^2/2}\, e^{-i\omega t}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega_0 t}\, e^{-\frac{a^2}{2}(\omega-\omega_0)^2 - i(\omega-\omega_0)t}$$

write the exponent as a perfect square

$$= \int_{-\infty}^{\infty} \frac{d(\omega-\omega_0)}{2\pi}\, e^{-i\omega_0 t}\, e^{-\frac{a^2}{2}\left(\omega-\omega_0+i\frac{t}{a^2}\right)^2}\, e^{-t^2/2a^2}$$

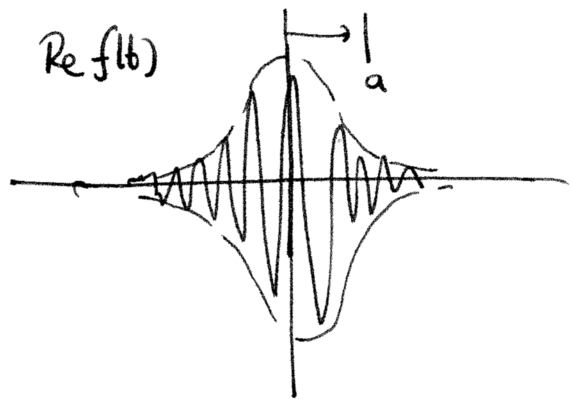
Now use

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{-\frac{a^2}{2}x^2} = \frac{1}{\sqrt{2\pi a^2}}$$

to complete the integral:

$$f(t) = \frac{1}{\sqrt{2\pi a^2}}\, e^{-i\omega_0 t}\, e^{-t^2/2a^2}$$

so this is a "packet", centered on $t=0$, of width $\Delta t = a$, containing an oscillation at ω_0



Following the same analysis with the wavepacket

$$\tilde{f}_{t_0}(\omega) = e^{-a^2(\omega-\omega_0)^2/2}\, e^{i\omega t_0}$$

we find that the real space form is

$$\int \frac{d\omega}{2\pi}\, \tilde{f}_{t_0}(\omega)\, e^{-i\omega t} = f(t-t_0)$$

the Gaussian wavepacket of width a centered at $t=t_0$. By dividing a real-space signal into wave packets, we can see how it translates into a shape in frequency space.

Let's look once more at the formula for $I(t)$ on p. 6 and find another interpretation of this formula. Start with

$$I(t) = \int \frac{d\omega}{2\pi}\, \frac{\tilde{V}(\omega)}{Z(\omega)}\, e^{-i\omega t}$$

and introduce the inverse transforms

$$\tilde{V}(\omega) = \int dt \; V(t)\, e^{i\omega t}$$

$$\frac{1}{Z(\omega)} = \int dt \; \mathcal{Z}(t)\, e^{i\omega t}$$

then

$$I(t) = \int \frac{d\omega}{2\pi} \int dt_1 \, V(t_1) \int dt_2 \, \mathcal{Z}(t_2) \; e^{i\omega(t_1+t_2-t)}$$

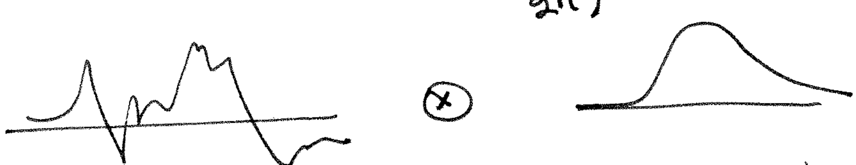
$$= \int dt_1 \, dt_2 \;\; \mathcal{Z}(t_2)\, V(t_1) \;\; \delta(t_1+t_2-t)$$

we can use the δ-function to do the t_2 integral and obtain:

$$I(t) = \int dt_1 \;\; \mathcal{Z}(t-t_1)\, V(t_1)$$

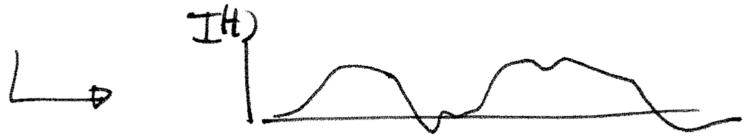
This structure is called a "convolution". The signal $V(t_1)$ is smeared out by the response of the circuit, encoded as $\mathcal{Z}(t)$. This function is, well, whatever has the Fourier transform $\frac{1}{Z(\omega)}$. Roughly, the convolution process looks like

$V(t)$ $\otimes$ $\mathcal{Z}(t)$



$\otimes$ ← sign for convolution

$I(t)$



The result we have just derived is a special case of the "convolution theorem":

Given functions $f(t)$, $g(t)$ with Fourier transforms $\hat{f}(\omega)$, $\hat{g}(\omega)$ the function with Fourier transform $\hat{f}(\omega)\cdot\hat{g}(\omega)$ is

$$F(t) = f \otimes g\,(t) = \int_{-\infty}^{\infty} dt'\; f(t-t')\, g(t')$$

"the convolution of f with g".

Very typically, the response of a linear device is the <u>convolution</u> of the original signal with a "response function" ($\mathfrak{z}(t)$ in the example above).