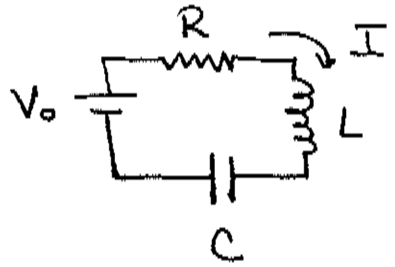
Circuits and Impedance

Jan. 12

At the end of the last lecture, we studied an ideal inductor as a circuit element with the behavior

$$V = L \frac{dI}{dt}$$

with the sign resisting a change in I. We saw that, in a circuit with a resistor, an inductor slows the transition to the steady-state current. In a circuit with a capacitor, we have a more interesting behavior. Consider



$$V_0 = \frac{Q}{C} + IR + L\frac{dI}{dt}$$

now $I = \frac{dQ}{dt}$, so we can rewrite this equation as

$$V_0 = \frac{Q}{C} + R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2}$$

In the steady-state behavior,

$$Q = CV_0 \qquad I = 0$$

and the capacitor is completely charged. But, if we start with an open circuit and close it, how do we get there?

Rewrite the equation as

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{1}{L}V_0$$

This equation has exactly the form of a harmonic oscillator equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = \frac{F(t)}{m}$$

With $\gamma = 0$, the homogeneous equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

has solutions $$x = A\cos(\omega t + \phi)$$

ω is the frequency of the oscillation. If γ is small, this term supplies some small damping

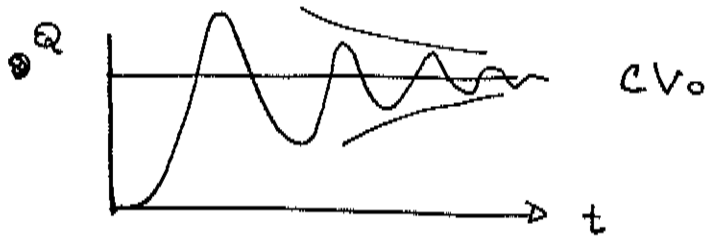
$$x(t) = Ae^{-\gamma t}\cos(\bar{\omega}t + \phi) \qquad \bar{\omega} \approx \omega$$

[I'll derive this result in a moment.] We have a second-order differential equation and this solution has two arbitrary parameters, so it is the most general solution. To solve the full problem, add one solution to the inhomogeneous equation.

Then $$Q(t) = CV_0 + Ae^{-\gamma t}\cos(\bar{\omega}t + \phi)$$

If we start with $Q=0$ $I=0$ at $t=0$,

$$Q(t) \simeq CV_o\,(1 - e^{-\gamma t}\cos\bar{\omega} t)$$



with $\qquad \omega = \dfrac{1}{\sqrt{LC}} \qquad T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{LC}$

units. $\quad (H\cdot F)^{1/2} = \left[\dfrac{V\cdot sec}{A}\cdot\dfrac{C}{V}\right]^{1/2} = [sec^2]^{1/2}$
$= sec$!

(for a $1\,\mu H$ inductor and $1\,\mu F$ capacitor,
$T \sim 10^{-6}$ sec.)

and damping time: $\qquad \dfrac{1}{\gamma} = \dfrac{L}{R}$

For R small, $\gamma \ll \omega$ or $\frac{1}{\gamma} \gg T$, the circuit then exhibits oscillatory behavior; we say that it is <u>resonant</u>.

I'd now like to give two derivations of the damping time, each of which illustrates a useful method. First, study the undamped oscillator and introduce γ as a small perturbation. A useful way to represent the freely oscillating solution is to write

$$x = \text{Re}\,[A e^{-i\omega t}]$$

The equation for $x(t)$ is linear with real coefficients, so if $\mathbb{X} = a e^{-i\omega t}$ solves the equation:

$$\frac{d^2\mathbb{X}}{dt^2} + \omega^2 \mathbb{X} = 0$$

then we can take the real part and find that $x = \mathrm{Re}[a e^{-i\omega t}]$ solves the real-valued equation we had originally. Write $a = A e^{-i\phi}$, we then recover $x = A\cos(\omega t + \phi)$

The complex notation gives a nice way to discuss the energy in the oscillator. The harmonic oscillator equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

has an associated conserved energy

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} m\omega^2 x^2$$

[Proof: $\frac{dE}{dt} = m\frac{dx}{dt}\cdot\left(\frac{d^2x}{dt^2} + \omega^2 x\right) = 0$]

The second term is the potential energy. Plugging in our solution, this equals

$$\frac{1}{2} m\omega^2 x^2(t) = \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi)$$

The average over a cycle (or over many cycles) is

$$\left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle = \frac{1}{2} m\omega^2 \cdot \frac{1}{2} A^2 = \frac{1}{2} m\omega^2 \cdot \frac{1}{2} |a|^2$$

The first term is the kinetic energy. Plugging in our solution, this equals:

$$\tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \tfrac{1}{2}m\ A^2\omega^2 \sin^2(\omega t+\phi)$$

The average of this over many cycles is

$$\left\langle \tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2\right\rangle = \tfrac{1}{2}m\cdot\tfrac{1}{2}A^2\omega^2 = \tfrac{1}{2}m\ \tfrac{1}{2}\omega^2|a|^2$$

$$= \tfrac{1}{2}m\cdot\ \tfrac{1}{2}\left|\frac{d}{dt}(ae^{-i\omega t})\right|^2$$

so we see that the representation of an oscillation as the real part of a complex exponential is very convenient for computing time averages: we just compute

$$\left\langle [\mathrm{Re}\ (ae^{-i\omega t})]^2\right\rangle = \tfrac{1}{2}|a|^2$$

Now we can use this result to interpret γ. The total energy of the oscillator is:

$$\text{kinetic} + \text{potential} = \tfrac{1}{2}m\omega^2|a|^2$$

Now add in the γ term:

$$\frac{d}{dt}E = \frac{d}{dt}\left(\tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}m\omega^2x^2\right)$$

$$= m\frac{dx}{dt}\left(\frac{d^2x}{dt^2}+\omega^2x\right)$$

$$= m\frac{dx}{dt}\left(-2\gamma\frac{dx}{dt}\right)$$

so $$\frac{dE}{dt} = -2m\gamma\left(\frac{dx}{dt}\right)^2$$

averaging over many cycles

$$\left\langle \frac{dE}{dt} \right\rangle = -2m\gamma \left\langle \left(\frac{dx}{dt}\right)^2 \right\rangle = -2m\gamma \cdot \tfrac{1}{2}\omega^2 |a|^2$$

$$\tfrac{1}{2} m\omega^2 \frac{d}{dt}|a|^2 = \tfrac{1}{2} m\omega^2 \cdot (-2\gamma |a|^2)$$

so indeed $\quad |a|^2(t) \cong e^{-2\gamma t} |a|^2(0) \quad$ or $\quad A(t) \cong e^{-\gamma t} A$

This semiautomatic way of using a complex representation to average over oscillations will be very useful when we study the properties of waves.

Returning to the circuit problem,

$$L \frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

the conserved energy for $R = 0$ is

$$E = \tfrac{1}{2} L \left(\frac{dQ}{dt}\right)^2 + \tfrac{1}{2}\frac{Q^2}{C}$$

$$= \tfrac{1}{2} L I^2 + \tfrac{1}{2} Q^2/C$$

we recognize the second term — the one that plays the role of potential energy — as the energy stored in the capacitor. The first term is the energy stored as magnetic field in the inductor. It is not so hard to connect this to our previous formulae:

$$L_s I = (\text{flux thgh the circuit}) = \int d\vec{l}\cdot\vec{A}$$

$$\frac{1}{2} L_s I^2 = \frac{1}{2} I \int d\vec{l}\cdot\vec{A} = \frac{1}{2}\int d^3y\ \vec{J}\cdot\vec{A}$$

(see last term for this step) $$= \frac{1}{2\mu_0}\int d^3y\ B^2$$

I promised you two derivations of the relation between γ and the oscillator damping. The first was only approximate. What if we want to be more exact? If we go back to the full equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = 0$$

we can find the exact solution of this equation by noting that, if complex $X(t)$ solves the equation, so does $\mathrm{Re}[X(t)] = x(t)$.

So try

$$X(t) = a\, e^{-i\Omega t}$$

$$0 = \frac{d^2X}{dt} + 2\gamma\frac{dX}{dt} + \omega^2 X = a e^{-i\Omega t}\left[(-i\Omega)^2 + 2\gamma(-i\Omega) + \omega^2\right]$$

So the solution of the equation reduces to an algebraic equation for Ω:

$$-\Omega^2 - i\,2\gamma\Omega + \omega^2 = 0$$

$$\Omega = -i\gamma \pm [\omega^2 - \gamma^2]^{\frac{1}{2}}$$

Choosing the + solution, we find

$$X(t) = a\, e^{-\gamma t} e^{-i\bar{\omega} t} \qquad \bar{\omega} = [\omega^2 - \gamma^2]^{\frac{1}{2}}$$

$$\cong \omega - \tfrac{1}{2}\gamma^2/\omega + \cdots$$

$$x(t) = A\, e^{-\gamma t} \cos(\bar{\omega} t + \phi)$$

as claimed above. This has two parameters, so it is the most general solution to the second-order differential equation. If we have chosen the $-$ solution, we would have found the same family of $x(t)$.

This method of turning a real-valued linear differential equation into a complex-valued differential equation is very effective. (Of course, it only works for linear equations.) We can use it, for example, to analyze the effect of an external force or — in the electrical problem — an external voltage. Go back to the original circuit problem

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = V$$

and consider the effect of an oscillating applied voltage

$$V(t) = V_0 \cos \Omega t$$

It is much easier to solve the equation if we complexify it and write

$$V(t) = \mathrm{Re}[V_0 e^{-i\Omega t}]$$

Then $Q = \mathrm{Re}\, Q_0 e^{-i\Omega t}$, where

$$\left(L(-i\Omega)^2 + R(-i\Omega) + \tfrac{1}{C}\right) Q_0 e^{-i\Omega t} = V_0 e^{-i\Omega t}$$

Then

$$Q_0 = \frac{V_0}{\frac{1}{C} - i\Omega R - \Omega^2 L}$$

$$= \frac{CV_0}{1-\Omega^2 LC - i\Omega RC}$$

Using the time averaging method discussed above, the energy stored in the capacitor has a time average:

$$\frac{1}{2}\frac{1}{C}\cdot\left|\frac{Q_0}{2}\right|^2 = \frac{1}{4}V_0^2 C \frac{1}{(1-\Omega^2 LC)^2+(\Omega RC)^2}$$

and the energy stored in the inductor is

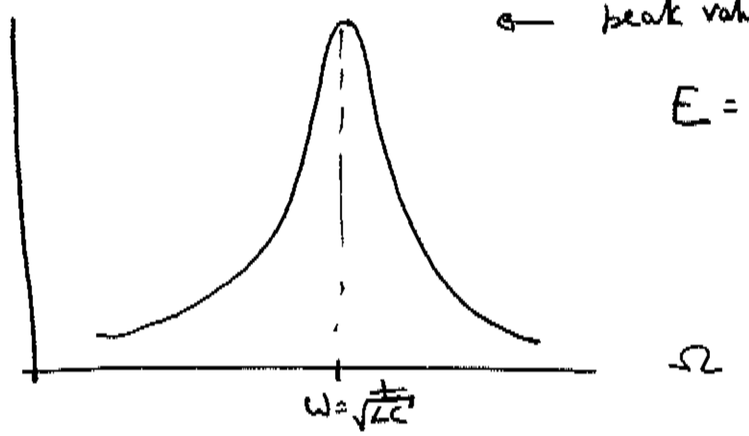
$$\frac{1}{2}L\left|\frac{-i\Omega Q_0}{2}\right|^2 = \frac{1}{4}L\Omega^2 C^2 V_0 \frac{1}{(1-\Omega^2 LC)^2+(\Omega RC)^2}$$

so the time-average energy in the whole circuit is:

$$\langle E\rangle = \frac{1}{2}CV_0^2 \cdot \frac{1}{2}(1+\Omega^2 LC)\frac{1}{(1-\Omega^2 LC)^2+(\Omega RC)^2}$$

As you might expect, this peaks up very strongly at the natural frequency

$$\Omega = \omega = \frac{1}{\sqrt{LC}}$$



← peak value:

$$E = \frac{1}{2}\frac{V_0^2}{R^2/L}$$

and, as the damping is removed, $R \to 0$, the peak energy stored $\to \infty$.

There is a simple way to express this algebraic method for solving circuit problems. Canonically, instead of using Q, one uses I, writing (I also replace $\Omega \to \omega$)

$$I(t) = \mathrm{Re}\, I_0 e^{-i\omega t} \qquad Q(t) = \mathrm{Re}\left[\left(\frac{I_0}{-i\omega}\right) e^{-i\omega t}\right]$$

Then our problem has the form

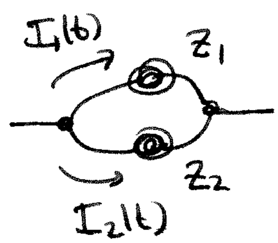
$$L(-i\omega)I_0 + RI_0 + \frac{1}{C}\left(\frac{1}{-i\omega}\right)I_0 = V_0$$

This looks exactly like Ohm's law, but with complex circuit elements. Each circuit element has the form of a resistor, and the equation above is the relation for resistors added in series. Thus, it makes sense to define

$$\underline{\text{impedance}}\ Z = \begin{cases} R & \text{for a resistor} \\ i/\omega C & \text{for a capacitor} \\ -i\omega L & \text{for an inductor} \end{cases}$$

[Note: engineers write $-i = j$ in these formulae.]

For a circuit driven at the AC frequency ω. To analyze the circuit, add impedances for devices in series. For devices in parallel



$I_1(t) = \mathrm{Re}\, I_{10}\, e^{-i\omega t}$ $\qquad$ $I_2(t) = \mathrm{Re}\, I_{20}\, e^{-i\omega t}$

voltage across the combination $= V(t) = \mathrm{Re}\, V_0\, e^{-i\omega t}$

$I_{10} = \dfrac{V_0}{Z_1}$, $I_{20} = \dfrac{V_0}{Z_2}$, and so the total current

is $\quad (I_1 + I_2)(t) = \mathrm{Re}\left[(I_{10} + I_{20})\, e^{-i\omega t}\right]$

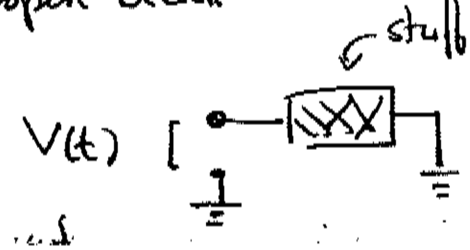
$$= \mathrm{Re}\left(\frac{V_0}{Z_1} + \frac{V_0}{Z_2}\right) e^{-i\omega t}$$

so the effective impedance of devices in parallel is

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

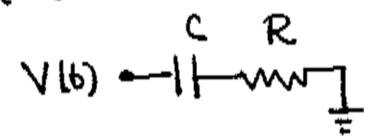
so the rules for combining impedances are precisely the usual rules for combining resistors.

Here are some examples. In each case, I'll analyze an open circuit:



← ground: $\phi = 0$.

Consider first



$$I_0 = \frac{V_0}{R + i/\omega C}$$

The voltage drop across the resistor is given by

$$V_{R0} = \frac{V_0 R}{R + i/\omega C} \qquad V_R(t) = \mathrm{Re}\left[V_{R0}\, e^{-i\omega t}\right]$$

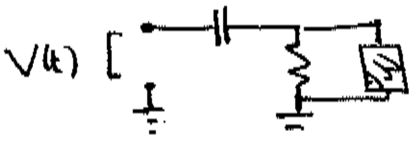
$$\left|\frac{V_{R0}}{V_0}\right| = \left[\frac{R^2}{R^2 + \frac{1}{\omega^2 C^2}}\right]^{\frac{1}{2}} = \left(\frac{\omega^2 R^2 C^2}{1 + \omega^2 R^2 C^2}\right)^{\frac{1}{2}}$$

The voltage drop across the capacitor is

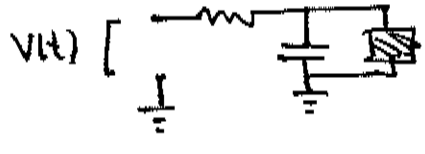
$$V_{C0} = \frac{V_0 \cdot i/\omega C}{R + i/\omega C}$$

$$\left|\frac{V_{C0}}{V_0}\right| = \left[\frac{1}{1 + \omega^2 R^2 C^2}\right]^{\frac{1}{2}}$$

So the high-frequency response is mainly across the resistor, the low-frequency response mainly across the capacitor. The dividing line is $\omega \sim 1/RC \sim 1/\tau$, as we might expect. If we use this circuit to power a device with ~~high~~ impedance

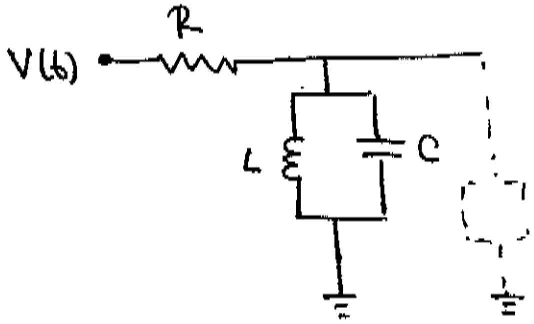


sends only the high-frequency oscillations to the device ("high-pass filter")



sends only the low-frequency oscillations to the device ("low-pass filter")

Next, consider the circuit



The impedance of the LC combination is

$$Z_{LC} = \left[\left(\frac{1}{-i\omega L}\right) + \frac{\omega C}{i}\right]^{-1} = \left(\frac{-i\omega L}{1-\omega^2 LC}\right)$$

The current flowing through the circuit is given by

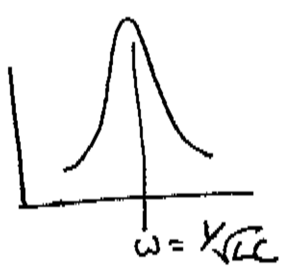
$$I_0 = \frac{V_0}{R - i\,\frac{\omega L}{1-\omega^2 LC}}$$

The voltage across the LC combination is

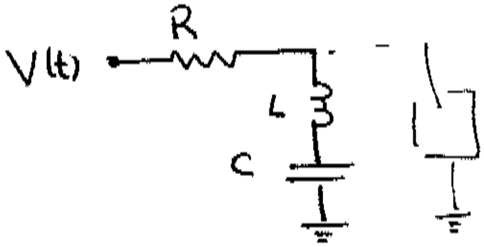
$$\frac{V_{LC0}}{V_0} = \left|\frac{\omega L}{\omega L + i\,R(1-\omega^2 LC)}\right|$$

$$= \left[\frac{\omega^2 L^2}{\omega^2 L^2 + R^2(1-\omega^2 LC)^2}\right]^{1/2}$$

which is resonant at $\omega = 1/\sqrt{LC}$



on the other hand, in



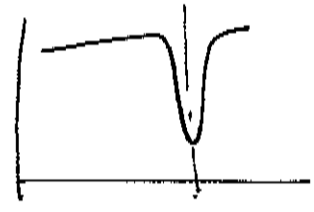
the LC combination has impedance $Z_{LC} = -i\omega L + \frac{i}{\omega C}$.

$$I_0 = \frac{V_0}{R - i\frac{(\omega^2 LC - 1)}{\omega C}}$$

$$\frac{V_{LC0}}{V_0} = \left| \frac{-i(\omega^2 LC - 1)}{R\omega C - i(\omega^2 LC - 1)} \right|$$

$$= \left[\frac{(\omega^2 LC - 1)^2}{R^2\omega^2 C^2 + (\omega^2 LC - 1)^2} \right]^{\frac{1}{2}}$$

this circuit <u>cuts out</u> the response near $\omega = \frac{1}{\sqrt{LC}}$



With more involved combinations of circuit elements, one can design a circuit with the desired pattern of response to various input frequencies.