March 12

Energy, Momentum, and Forces in Special Relativity

Now that we have seen how to think about Lorentz transformations geometrically, I would like to take a step in the other direction and develop their mathematical formalism.

Let $x = (x^0, x^1, x^2, x^3)$ and $x' = (x^{0\prime}, x^{1\prime}, x^{2\prime}, x^{3\prime})$ be related by a Lorentz transformation. Then we can consider x, x' as 4-dimensional vectors and the Lorentz transformation as a matrix acting on one vector to produce the other.

$$x = \underbrace{\Lambda}_{4\times 4 \text{ matrix}} x'$$

For example:

if x is measured in a frame F and x' is measured in a frame F' boosted by $\vec{V} = V\hat{z}$

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \qquad \beta = V/c \qquad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$$

· if x is measured in F and x' is measured in a frame F' boosted by $\vec{V} = V\hat{x}$

$$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

if x is measured in F and x' is measured in a frame F' rotated by θ about the $\hat{z}$ axis

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

I would like to define a 4-vector as a set of four physical quantities $(Q^0, Q^1, Q^2, Q^3) \equiv Q$ whose values depend on the frame, but in such a way that

Q measured in $F \longrightarrow Q_F$

Q measured in $F' \longrightarrow Q_{F'}$

$$Q_F = \Lambda \, Q_{F'}$$

To show you that this idea of a "4-vector" is useful, I would like to present several important physical quantities which are 4-vectors.

To begin, I will show that the velocity can be converted to a 4-vector. Consider a particle moving through space. Its location at time t is $\vec{x}(t)$. Now let τ be the proper time for the particle — the time read off on a clock carried with the particle. Time dilatation says:

$$dt = \frac{d\tau}{\sqrt{1-v^2/c^2}}$$

Now, since $x(t) = (t, \vec{x}(t))$ is a 4-vector and τ is a Lorentz invariant,

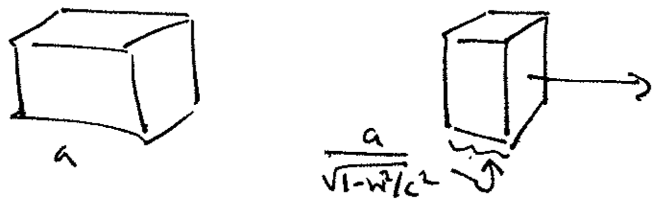
$$V = \frac{dx}{d\tau}$$

is also a 4-vector. Explicitly

$$V = \frac{dt}{d\tau}\frac{dx}{dt} = \left(\frac{c}{\sqrt{1-v^2/c^2}}, \frac{\vec{v}}{\sqrt{1-v^2/c^2}}\right)$$

Here is a more ~~interesting~~ example. Since a boost affects the velocity of a particle, the density and current of particles or of electric charge are affected by boosts. Let's start with a simple model in which we have a cloud of electrons, all with the same velocity. We can go to a frame where these electrons are at rest. Let the density of electrons in that frame be ρ_0.

Now boost to a frame travelling with velocity $-\vec{w}$, so that the electrons all acquire velocity $\vec{w}$. The density of electrons also increases, since a cube of side a in the original frame



appears Lorentz contracted in the new frame but still contains the same number of electrons. So in the new frame

$$\rho = \frac{\rho_0}{\sqrt{1-w^2/c^2}} \qquad \vec{j} = \frac{\rho_0}{\sqrt{1-w^2/c^2}}\,\vec{w} = \rho\vec{w}$$

Thus,

$$(c\rho, \vec{j}) = J = \rho_0 V$$

for this model, where ρ_0 is the density in the special frame and V is the velocity 4-vector of the electrons. A more general situation in which the electrons have a distribution of velocities can be viewed as a superposition of systems of this type. Thus,

$$J = (c\rho, \vec{j})$$

is a 4-vector

Now I would like to propose that the energy and momentum of a particle form a 4-vector, and that in fact

$$\mathcal{P} = (E/c, \vec{p}) = m\,V$$

so that

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}} \qquad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$$

This proposal passes a number of important tests:

(0) Since E is rotationally invariant and $\vec{p}$ is a vector, these expressions must be of the form:

$$E = f(v^2) \qquad \vec{p} = \vec{v}\, g(v^2)$$

(1) E and $\vec{p}$ have the correct limiting forms for small velocity

$$E = mc^2\left(1 + \tfrac{1}{2} v^2/c^2 + \cdots\right) \qquad \vec{p} = \underbrace{m\vec{v}}_{\text{correct Newtonian momentum}}\,(1+\cdots)$$

$$= (\text{const}) + \underbrace{\tfrac{1}{2}mv^2}_{\text{correct kinetic energy}} + \cdots$$

(2) E and p satisfy the usual relation between force and work:

$$\frac{dE}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt}$$

(you might question the necessity of this relation, but we will see that we need it to prove energy and momentum conservation in the coupling of matter to Maxwell's equations.)

Check the relation explicitly:

$$\frac{d\vec{p}}{dt} = \frac{m\, d\vec{v}/dt}{\sqrt{1-v^2/c^2}} + m\vec{v}\, \frac{\vec{v}\cdot d\vec{v}/dt\ \frac{1}{c^2}}{(1-v^2/c^2)^{3/2}}$$

$$= \frac{1}{(1-v^2/c^2)^{3/2}} \left[m\frac{d\vec{v}}{dt} - m\frac{d\vec{v}}{dt}\frac{v^2}{c^2} + m\vec{v}\, \frac{\vec{v}}{c^2}\cdot\frac{d\vec{v}}{dt} \right]$$

$$\vec{v}\cdot\frac{d\vec{p}}{dt} = \frac{1}{(1-v^2/c^2)^{3/2}}\, m\, \vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$\frac{dE}{dt} = \frac{mc^2}{(1-v^2/c^2)^{3/2}}\, m\, \frac{\vec{v}}{c^2}\cdot\frac{d\vec{v}}{dt} = \frac{m}{(1-v^2/c^2)^{3/2}}\, \vec{v}\cdot\frac{d\vec{v}}{dt} \quad \checkmark$$

(3) If E and $\vec{p}$ are observed to be conserved by one observer, they will be conserved to any observer.

That is, if $(E, \vec{p})$, $(E', \vec{p}\,')$ are measurements of energy and momentum in two different frames, for any particle collision

$$1 + 2 \longrightarrow 3 + 4$$

The statements

$$E_1' + E_2' = E_3' + E_4' \qquad \vec{p}_1\,' + \vec{p}_2\,' = \vec{p}_3\,' + \vec{p}_4\,'$$

should imply: $E_1 + E_2 = E_3 + E_4 \qquad \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$

If $(E/c, \vec{p}) = \mathbb{P}$ is a 4-vector, this is obvious:

$$\mathbb{P}_1' + \mathbb{P}_2' = \mathbb{P}_3' + \mathbb{P}_4'$$

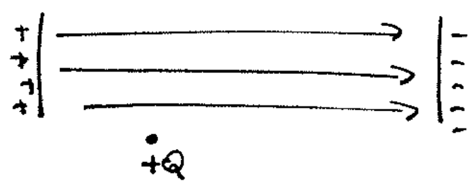
$$\Downarrow$$

$$\Lambda \mathbb{P}_1' + \Lambda \mathbb{P}_2' = \Lambda \mathbb{P}_3' + \Lambda \mathbb{P}_4'$$

which is just $\quad \mathbb{P}_1 + \mathbb{P}_2 = \mathbb{P}_3 + \mathbb{P}_4$

It can be shown that $\mathbb{P} = m\mathbb{V}$ is the unique set of expressions of the form (0) that satisfies this requirement. For an elementary (but not simple) proof, see Jackson (section 11.5 of the 2nd edition).

Armed with this understanding of energy and momentum, we can compute the motion of relativistic charged particles in electric fields. Consider first the situation of a constant electric field in the $\hat{z}$ direction



The equation of motion for a charge $+Q$ of mass m must be

$$\frac{d\vec{p}}{dt} = Q\vec{E} = QE\hat{z}$$

since we saw earlier in the course that this is the momentum bought by the electric field. This equation gives

$$\frac{d}{dt}\left(\frac{m\vec{v}}{1-v^2/c^2}\right) = QE\,\hat{z}$$

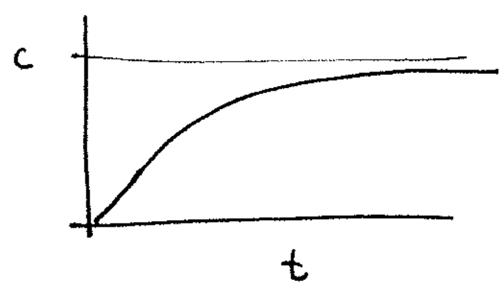
If the particle starts from rest

$$\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right) = QEt\,\hat{z}$$

so that

$$v = \frac{QEt/m}{\left[1+\left(\frac{QEt}{mc}\right)^2\right]^{1/2}}$$

This velocity begins with the Newtonian formula $v \approx \frac{QEt}{m}$ but asymptotically approaches $v \to c$



For this particle, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \left[\frac{1}{1-\frac{(QEt/mc)^2}{1+(QEt/mc)^2}}\right]^{1/2}$

or $$\gamma = \left[\frac{1+(QEt/mc)^2}{1}\right]^{1/2}$$

so that the energy of the particle is

$$E = \gamma mc^2 = mc^2 \left[1 + \left(\frac{QEt}{mc}\right)^2 \right]^{1/2}$$

For small t, this is

$$E = (\text{const}) + \frac{p^2}{2m} + \cdots$$

the Newtonian expression. For large t

$$E = (QEt) \cdot c \cong pc$$

Actually in general, for $v \to c$

$$E/c = \frac{c}{\sqrt{1-v^2/c^2}} \qquad p = \frac{v}{\sqrt{1-v^2/c^2}} \approx \frac{E}{c} \quad \text{as } v \to c$$

You can check explicitly what we already know from general principles, that $E(t)$ satisfies

$$\frac{dE}{dt} = \vec{v} \cdot Q\vec{E} = \vec{v} \cdot \frac{d\vec{p}}{dt}$$

This is the energy lost by the electric field according to Maxwell's equations.

Let's now look at this system from another frame.

let x', t' p', E' be measured in a frame F'

x t p E be measured in a frame F

such that F' is boosted with respect to F by $\beta\hat{z}$.

Then

$$t = \gamma (t' + \beta x^{3\prime})$$

so $$dt = \gamma (dt' + \beta\, dx^{3\prime}) = \gamma\, dt' \left(1 + \beta \frac{dx^{3\prime}}{dt'}\right)$$

$$p^3 = \gamma (p^{3\prime} + \beta E'/c)$$

so $$dp^3 = \gamma (dp^{3\prime} + \beta\, dE'/c)$$

similarly $$dp^1 = dp^{1\prime} \qquad dp^2 = dp^{2\prime}$$

If the particle is instantaneously at rest in F'

$$\frac{dx^{3\prime}}{dt'} = 0 \qquad \frac{dE'}{dt} = \vec{v}' \cdot \frac{d\vec{p}'}{dt} = 0$$

and we have $dt = \gamma\, dt'$ and so

$$\frac{dp^3}{dt} = \frac{dp^{3\prime}}{dt'} \qquad \frac{dp^1}{dt} = \frac{1}{\gamma}\frac{dp^{1\prime}}{dt'} \qquad \frac{dp^2}{dt} = \frac{1}{\gamma}\frac{dp^{2\prime}}{dt}$$

Then, if a particle at rest experiences that force $\vec{F}'$ in its rest frame:

$$\frac{d\vec{p}'}{dt'} = \vec{F}'$$

Then in a frame where the particle is moving with velocity v in the $\hat{3}$ direction

$$\frac{d\vec{P}}{dt} = \vec{F} \quad \text{, where} \quad F^1 = \frac{1}{\gamma} F^{1\prime} \qquad F^2 = \frac{1}{\gamma} F^{2\prime}$$

and $$F^3 = F^{3\prime}$$

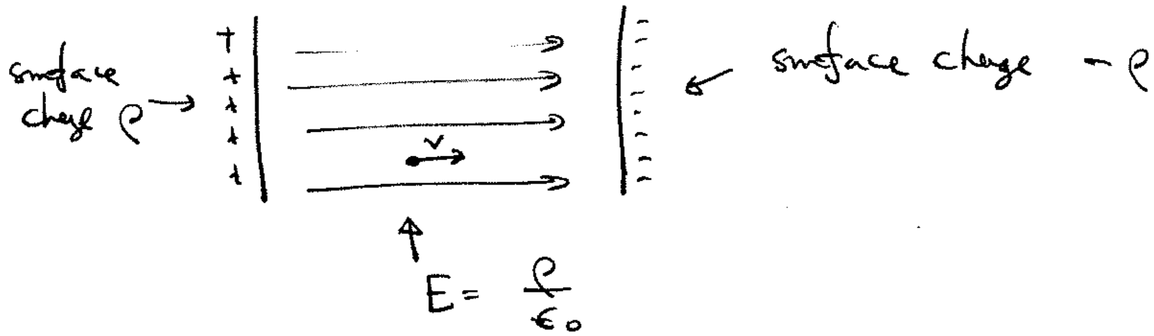
Notice that it gets harder and harder to deflect a particle as the velocity of the particle increases. On the other hand, for linear motion of the type we have just analyzed

$$\frac{dp}{dt} = \left(\frac{dp}{dt}\right)_{\text{rest frame}}$$

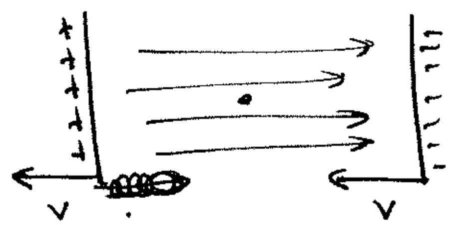
This makes good sense, because it implies in the example above

$$\vec{E}\Big|_{\text{original frame}} = \vec{E}\Big|_{\text{rest frame}}.$$

In the original frame we have:



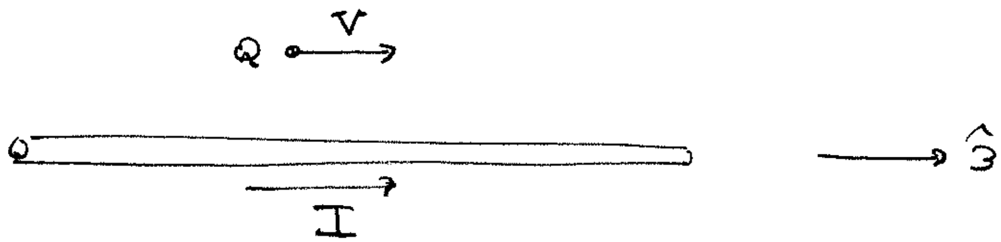
In the boosted frame in which the particle is at rest



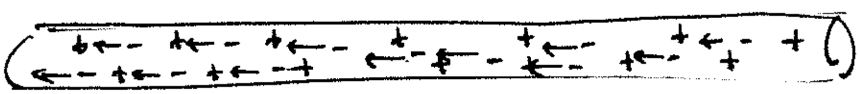
but the surface density of charge (C/m^2) is the same, so by Gauss' law, $\vec{E}$ should be the same.

This very simple relation for $\vec{E}$ applies only when we boost in the direction parallel to $\vec{E}$. Let's now consider some more general situations in which we boost distributions of charges and currents.

A very interesting situation is that of a wire carrying an electric current and a charge $+Q$ moving parallel to the wire, a situation analyzed very beautifully in Purcell's textbook:



Assume that the wire is electrically neutral, so that it produces no electrostatic field in this frame. As a model for this, consider the wire to contain electrons, of density ρ moving in the $-z$ direction with velocity $-\vec{W}$, and a neutralizing background of ions at rest with respect to the wire



$$I = e\rho|\vec{W}|$$

Analyze this system in the frame of the charge Q. In this frame, the background of positive ions is moving backward at velocity v. The density of positive ions is

$$\rho'_+ = \frac{1}{\sqrt{1-v^2/c^2}}\,\rho$$

The electrons move in this frame with velocity

$$w' = \frac{w+v}{1+wv/c^2}$$

The corresponding γ factor is

$$\frac{1}{\sqrt{1-w'^2/c^2}} = \frac{1}{\sqrt{1-w^2/c^2}}\,\frac{1}{\sqrt{1-v^2/c^2}}\left(1+{wv}/{c^2}\right)$$

The density of electrons in this frame is

$$\rho'_- = \frac{\sqrt{1-w^2/c^2}}{\sqrt{1-w'^2/c^2}}\,\rho = \frac{1}{\sqrt{1-v^2/c^2}}\left(1+{wv}/{c^2}\right)\rho$$

So, in the frame in which the charge Q is at rest, there is an excess of negative charge over positive charge on the wire.

$$\rho'_- - \rho'_+ = \frac{\rho}{\sqrt{1-v^2/c^2}}\,\frac{wv}{c^2}$$

this excess charge density sets up an electrostatic field

$$\vec{E}' = -\frac{1}{2\pi\epsilon_0}\,\frac{1}{r}\,\hat{r}\;\frac{e\rho}{\sqrt{1-v^2/c^2}}\,\frac{wv}{c^2}$$

pointing radially ~~inward~~, attracting the charge Q. The force on the charge Q is

$$\vec{F}' = -\hat{r}\,\frac{1}{2\pi\epsilon_0}\,\frac{1}{r}\,\frac{Q\cdot(e\rho w)\,v}{\sqrt{1-v^2/c^2}\;c^2}$$

$$= -\hat{r}\,\frac{1}{2\pi\epsilon_0 c^2}\,\frac{1}{r}\,\frac{1}{\sqrt{1-v^2/c^2}}\;QI\,v$$

Now transform back to the original frame. The force is $\perp$ to the direction of the boost, so

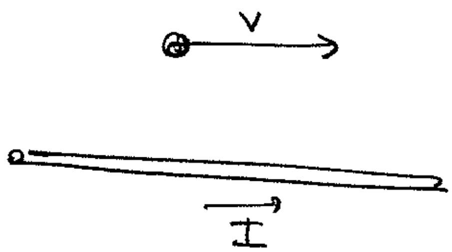
$$\vec{F} = \frac{1}{\gamma}\vec{F}' = \sqrt{1-v^2/c^2}\;\vec{F}'$$

$$\vec{F} = -\hat{r}\,\frac{QIv}{2\pi\epsilon_0 c^2}\,\frac{1}{r}$$

now $c^2 = \frac{1}{\epsilon_0\mu_0}$ so $\frac{1}{\epsilon_0 c^2} = \mu_0$

$$\vec{F} = (-\hat{r})\;Q\,v\cdot\left(\frac{\mu_0 I}{2\pi r}\right)$$

Thus, in a situation of a charged particle moving parallel to a current



the charge is attracted by a force

- proportional to Q, proportional to I, proportional to v

These are just the basic properties of the magnetic force that we studied last term. In fact we parametrized this force as

$$\vec{F} = Q\,\vec{v}\times\vec{B}$$

and assigned to the wire a B field

$$\vec{B} = \frac{\mu_0}{2\pi r} I\,\hat{\phi}$$

We have not rederived the force law only from electrostatics and the theory of relativity.