March 9

Theory of Relativity - 2

In the previous lecture, we considered the relation between $(t, \vec{x})$ measured in one inertial frame F and the corresponding coordinates $(t', \vec{x}')$ measured in an inertial frame F' moving with respect to F with constant velocity $\vec{v} = v\hat{z}$. We found that this relation has many weird properties.

Let's now try to find the explicit relation by a general analysis. Here are some reasonable requirements that the transformation $(t, \vec{x}) \to (t', \vec{x}')$ should satisfy:

① $x = x'$ $\quad y = y'$ (as we argued last time)

② (t', z') should be linear functions of (t, z)

③ for any choice of t, z, $\quad (ct')^2 - (z')^2 = (ct)^2 - z^2$

The constancy of the speed of light is the statement that

$$(x')^2 + (y')^2 + (z')^2 - (ct')^2 = 0 \Leftrightarrow x^2 + y^2 + z^2 - (ct)^2 = 0$$

Assumption ③ is a little stronger; basically it fixes the overall scale of the linear relation we are looking for.

To implement this, write $x^0 = ct$, $x^{0\prime} = ct'$, so that we put time intervals in units of meters. Then the general linear relation between (x^0, x^3) and $(x^{0\prime}, x^{3\prime})$ is:

$(z = x^3)$ $\qquad (z' = x^{3\prime})$

$$x^{0\prime} = A x^0 - B x^3$$
$$x^{3\prime} = -C x^0 + D x^3$$

then

$$(x^{0\prime})^2 - (x^{3\prime})^2 = (A^2 - C^2)(x^0)^2 - (D^2 - B^2)(x^3)^2 + (-2AB + 2CD)\, x^0 x^3$$

this $= (x^0)^2 - (x^3)^2$ for all possible (x^0, x^3) if

$$A^2 - C^2 = 1, \qquad D^2 - B^2 = 1 \quad \underline{\text{and}} \quad \frac{C}{A} = \frac{B}{D}$$

We have three equation for four unknowns, so there will be one free parameter. The first two equations are solved by writing

$$A = \cosh\eta \qquad C = \sinh\eta \qquad \text{for some hyperbolic angle } \eta$$
$$D = \cosh\zeta \qquad B = \sinh\zeta$$

The last equation implies $\tanh\eta = \tanh\zeta \Rightarrow \eta = \zeta$

Now define β by

$$\beta = \tanh\eta = \frac{\sinh\eta}{\cosh\eta}$$

Then

$$\cosh\eta = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma \qquad \sinh\eta = \frac{\beta}{\sqrt{1-\beta^2}} = \beta\gamma$$

The final form of the transformation is:

$$x^{0\prime} = \gamma(x^0 - \beta x^3)$$
$$x^{3\prime} = \gamma(x^3 - \beta x^0) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Solving for x^0, x^3 :

$$x^0 = \gamma(x^{0\prime} + \beta x^{3\prime})$$

$$x^3 = \gamma(x^{3\prime} + \beta x^{0\prime})$$

Let's now try to interpret these equations by looking at some special cases. First, look at the location of the origin of coordinates of F' as a function of time

$$x^{3\prime} = 0 \quad \Rightarrow \quad x^3 - \beta x^0 = 0$$

$$x^3 - (\beta c)t = 0$$

so this point moves in F in the $+\hat{z}$ direction a velocity $v = \beta c$. Thus,

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Next, consider a clock at rest at a fixed value of $x^{3\prime}$, say $x^{3\prime} = 0$. Then, if $t=0$ and $t'=0$ coincide,

$$x^0 = \gamma\, x^{0\prime} \quad \text{or} \quad t = \frac{t'}{\sqrt{1-v^2/c^2}}$$

so t', the time read on the clock in F', is smaller than t, the time elapsed in F, by a factor $\sqrt{1-v^2/c^2}$. Similarly, at a fixed value of x^3, say $x^3 = 0$,

$$x^{0\prime} = \gamma x^0 \quad \text{or} \quad t' = \frac{t}{\sqrt{1-v^2/c^2}}$$

so each observer thinks that the other's clocks run slow by the required time dilatation factor.

Next, let F put a stick of length L_1 along the $\hat{z}$ or $\hat{3}$ axis. The ends of the stick are at

$$(x^0, x^3) = (ct, 0) \qquad x^0, x^3 = (ct, L_1)$$

Since the stick is moving in F's frame, he must locate the ends simultaneously if one end is at

$$(x^{0\prime}, x^{3\prime}) = (0, 0)$$

the other will be at $(x^{0\prime}, x^{3\prime}) = (0, L_1')$ Then

$$x^3 = L_1 = \gamma(L_1' + \beta \cdot 0)$$

or

$$L_1' = \left(\frac{1}{\gamma}\right) L_1 = \sqrt{1-v^2/c^2}\; L_1$$

F' sees this stick as contracted. F sees the same effect for a stick at rest with respect to F'. Finally, look at the equation

$$x^{0\prime} = \gamma(x^0 - \beta x^3)$$

or

$$t' = \frac{1}{\sqrt{1-v^2/c^2}}\left(t - \frac{v}{c^2} x^3\right)$$

For events which occur at the same t, they are earlier in t' the further ahead they are. So our general analysis verifies all of the observations made in the previous lecture.

The transformation from (x^0, x^3) to $(x^{0\prime}, x^{3\prime})$ is called a Lorentz transformation or, more simply, a boost. The result of two successive boosts is a boost: If

$$x^{0\prime} = \gamma\,(x^0 - \beta x^3) \qquad\qquad x^{0\prime\prime} = \gamma'(x^{0\prime} - \beta x^{3\prime})$$
$$x^{3\prime} = \gamma\,(x^3 - \beta x^0) \qquad\qquad x^{3\prime\prime} = \gamma'(x^{3\prime} - \beta x^{0\prime})$$

then

$$x^{0\prime\prime} = \Gamma\,(x^0 - B\,x^3)$$
$$x^{3\prime\prime} = \Gamma\,(x^3 - B\,x^0)$$

where:
$$\Gamma = \gamma'\gamma\,(1+\beta\beta') \qquad B = \frac{\beta+\beta'}{1+\beta\beta'}$$

Interpreting $B = V/c$, where V is the velocity of the composite boost

$$V = \frac{v+v'}{(1+vv'/c^2)}$$

and one can show that
$$\Gamma = 1\Big/\sqrt{1-V^2/c^2}\,.$$

Notice ~~that~~ it is not possible, by composing boosts, to obtain a velocity faster than c! If we go back to

$$\beta = \tanh\eta \qquad \gamma = \cosh\eta$$

the composite of two boosts yields

$$\Gamma = \cosh\eta\cosh\eta + \sinh\eta\sinh\eta' = \cosh(\eta+\eta')$$

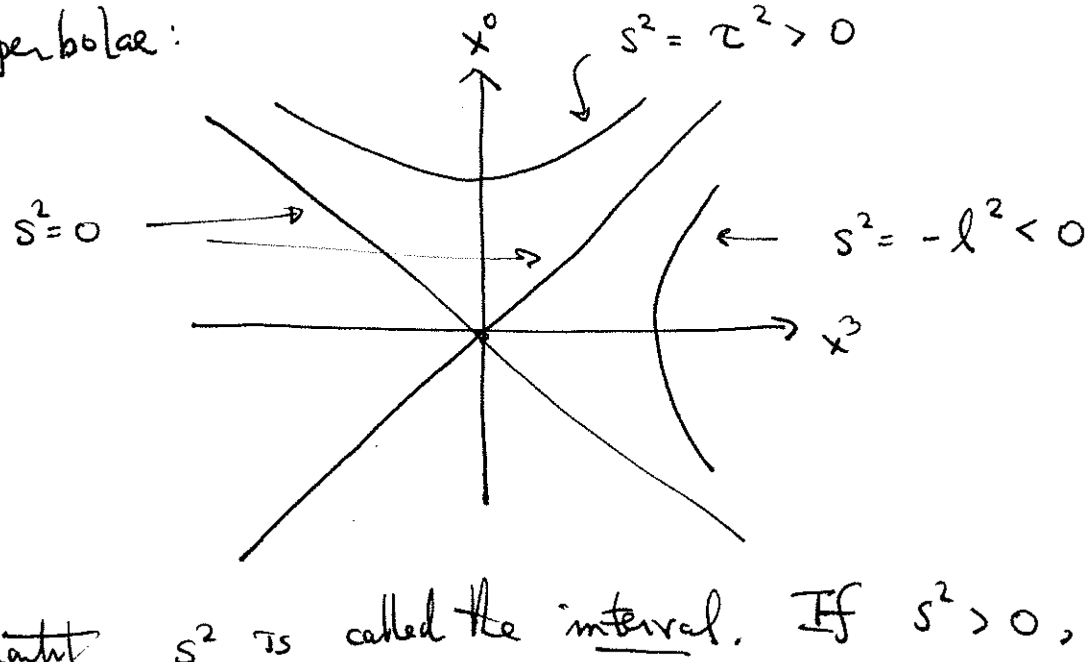
$$\Gamma B = \cosh\eta\sinh\eta' + \sinh\eta\cosh\eta' = \sinh(\eta+\eta')$$

so the hyperbolic angles η just add. The useful angle η is called the <u>rapidity</u> of the boost.

Let's look at these relations more geometrically. In the x^0, x^3 plane, the curves of constant

$$s^2 = (x^0)^2 - (x^3)^2$$

are hyperbolae:

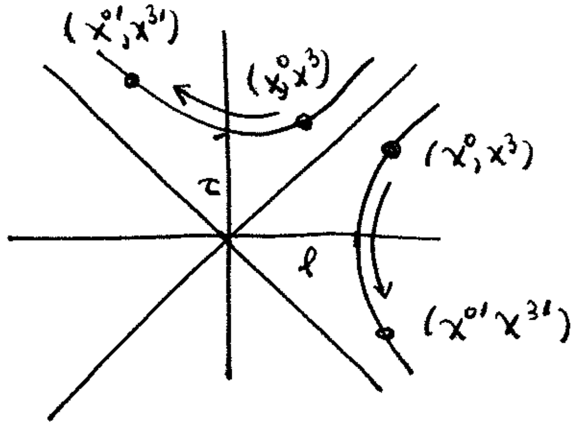


The quantity s^2 is called the <u>interval</u>. If $s^2 > 0$, we say that the interval (between (x^0, x^3) and $(0,0)$) is <u>time-like</u>. The value $\tau = \sqrt{s^2}$ is the time that elapses on a clock in the frame where $x^{3\prime} = 0$. This is called the "proper time" of the interval.

If $s^2 < 0$, we say that the interval is space-like. The value $l^2 = \sqrt{-s^2}$ is the length of the interval in the frame where $x^{0\prime} = 0$. This is called the "proper length" of the interval.

If $s^2 = 0$, we say that the interval is light-like. Then $x^0 = x^3$ or $x^3 = ct$ and the interval can be spanned by a light ray.

A boost preserves the value of s^2 and thus moves (x^0, x^3) to another point $(x^{0\prime}, x^{3\prime})$ on the same hyperbola



Notice that, if the interval between $(0,0)$ and $(x^0, x^3) = x$ is spacelike, the vector from $(0,0)$ to x goes forward in time in some frames and backwards in time in other frames.

Now generalize to

$$x = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

A more general boost which relates coordinates x in a frame F to coordinates x' in a Frame F', moving at velocity $\vec{v}$ with respect to F, is

$$x^{0\prime} = \gamma\left(x^0 - \frac{\vec{v}}{c}\cdot\vec{x}\right) \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

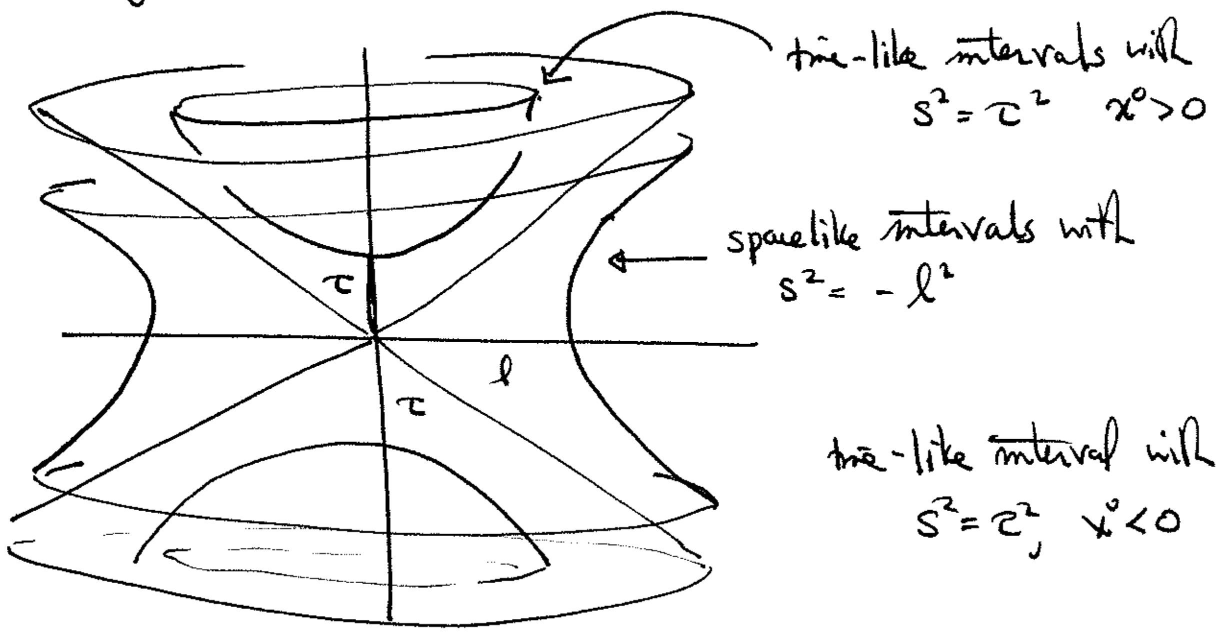
$$\hat{v}\cdot\vec{x}' = \gamma\left(\hat{v}\cdot\vec{x} - \left|\frac{\vec{v}}{c}\right| x^0\right)$$

and $(\vec{x}')_\perp = (\vec{x})_\perp$ for components perpendicular to $\vec{v}$

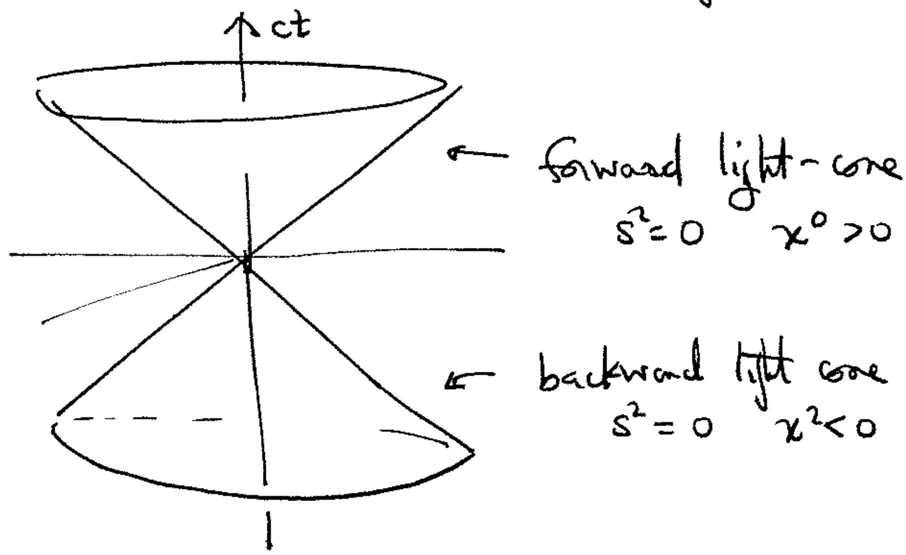
these transformation preserve the interval

$$s^2 = (ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

The surfaces of constant s^2 are hyperboloids:



The time-like and space-like regions are separated by two cones, the forward and backward light-cones.



Points inside the forward light-cone are definitely in the future of $(0,0)$. Points inside the backward light cone are definitely in the past of $(0,\vec{0})$. For points x outside the light cones, the time relation between x and $(0,\vec{0})$ is ambiguous, depending on the frame.

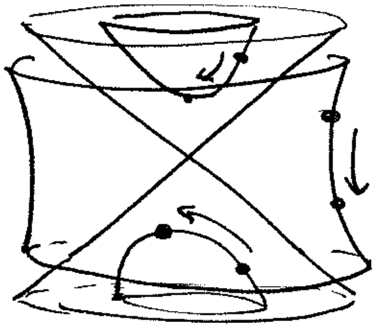
The linear transformations of (x^0, x^1, x^2, x^3) that preserve s are:

① boosts, parametrized by a velocity $\vec{v}$
② rotations of (x^1, x^2, x^3), parametrized by an angle $\vec{\alpha}$.
③ any transformation that can be built as a product of boosts and rotations.

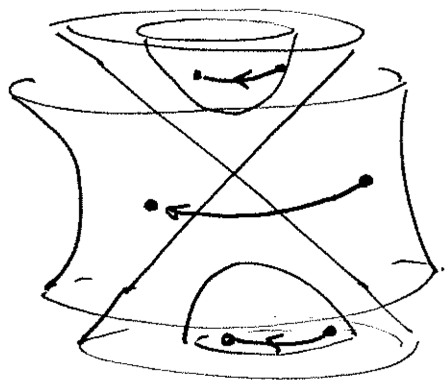
These transformations form a 6-parameter group of transformations called the Lorentz group.

Typical Lorentz group transformations are:

boost:



rotation:



notice that the surface $s^2 = -\ell^2$ has only one connected piece; any point on this surface is connect to any other point by a continuous Lorentz transformation. However, the surface $s^2 = \tau^2 \geq 0$ splits into two disconnected parts — the future and past hyperboloids.