Theory of Relativity

March 7

Up to this point in the course, we have treated the $\vec{E}$ and $\vec{B}$ fields as distinctly different. It is true that they are mixed up by the equations of motion, but we have treated them as different fields with different origin. In these last few lectures, I would like to explain that the $\vec{E}$ and $\vec{B}$ fields are actually two parts of the same fundamental object. In a similar way, we will find that other quantities that we have treated as distinct — ρ and $\vec{j}$ or $\mathcal{E}$, $j_\mathcal{E}$, $\vec{P}$ and T^{ij} — assemble into a larger and more fundamental object. These are all consequences of the special theory of relativity.

Einstein discovered the theory of relativity by thinking about Maxwell's equations. In fact, his first paper on relativity was titled "On the Electrodynamics of Moving Bodies". What was the problem, and how did he solve it?

It is a consequence of Maxwell's equations that electromagnetic waves move at the speed $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. This is, if you think about it, an unusual conclusion. Waves on water, on a string, or in an elastic medium move at a definite speed relative to the medium. When we analyzed such waves, we tacitly assumed ~~that~~ the medium (the string, the elastic body) was a rest. And we know that, if a wavepacket is moving on a string at speed v

and we run alongside the string, we can run right beside this wavepacket, and it will appear to be stationary with respect to us.

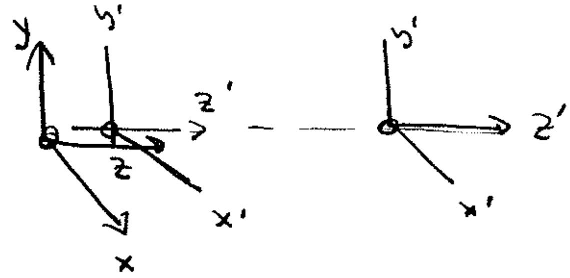
Newtonian physics contains the notion of an inertial frame of reference, a coordinate system in which a body at rest tends to remain at rest and a body moving at constant velocity does not accelerate. Though Newton believed in an absolute preferred coordinate system, two systems in uniform relative motion show the same laws of Nature. Let

$$(t,x,y,z) = (t,\vec{x})$$

be coordinates of an inertial frame (F), and let

$$(t',x',y',z')$$

be another coordinate system (F') that is moving in the $\hat{z}$ direction with respect to the first at velocity v.



For Newton, time was absolute: $t' = t$. The position of an object in the F frame would be related to the position of an object in the F' frame by

$$x' = x \qquad y' = y \qquad z' = z - Vt$$

If a particle is accelerated by a force

$$\frac{d^2}{dt^2}\vec{x}(t) = \frac{\vec{F}}{m}$$

then, since $\vec{x}' = \vec{x} - \vec{V}t$ $\qquad \vec{V} = V\hat{z}$ $\qquad$ "Gallilean relativity"

$$\frac{d^2}{dt^2}\vec{x}' = \frac{\vec{F}}{m}$$

and so, in the x' frame, the particle feels the same force.

However, an object that is moving with velocity $\vec{V}$ with respect to the F frame.

$$\vec{x}(t) = \vec{x}_0 + \vec{V}t$$

moves with respect to the F' at velocity $(\vec{V} - \vec{v})$

$$\vec{x}'(t) = \vec{x}_0 + (\vec{V} - \vec{v})t$$

for any velocity a body might have, there is always an inertial frame in which it is at rest.

From these considerations, it might seem that the conclusion of Maxwell's equations that electromagnetic waves propagate in every direction at speed c could only be true in one inertial frame. Maxwell would have been very comfortable with this conclusion, since he thought of electromagnetic waves as the vibrations of a medium, the "luminiferous ether". Maxwell's equations would be correct in the frame at rest with respect to the ether. However, from this point of view, it is odd that many of the consequences of Maxwell's equations are independent of the frame of reference. For example, Faraday's law of induction is correct whether the current loop is moving with respect to the magnetic field or vice versa. Also, many attempts at the end of the nineteenth century to measure the change in the speed of light in a moving frame (notably, the Michelson-Morley experiment) gave negative results. Einstein did not believe in the "ether", he felt that Maxwell's equations should describe fields in otherwise empty space. So he set about building a theory based on two postulates which, as he says in his 1905 paper, "are" only apparently irreconcilable":

1. The laws of physics are the same in all inertial frames of reference
2. Light always propagates at speed c, in every frame, independently of the state of motion of the emitter.

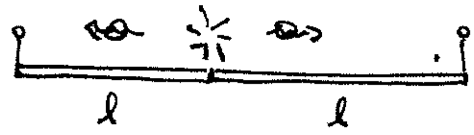
Now, if both postulates are right, the Galilean transformation law

$$t' = t \qquad z' = z - vt$$

must be wrong. What replaces it?

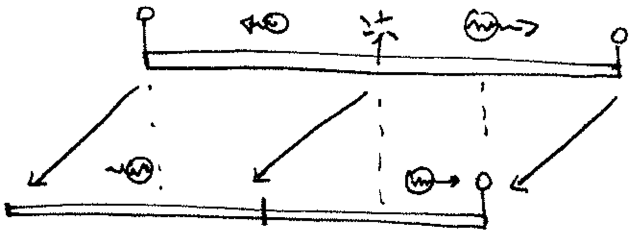
Einstein's amazing insight was that it was permissible for observers in F and F' to disagree on the definition of time. In particular, events that are simultaneous with respect to F need not be simultaneous with respect to F'! In fact, they cannot be, in general, if the speed of light is to be constant.

Here is a simple example. Let the emission of a flash of light at $x = x' = 0$ signal $t = 0$, $t' = 0$. Let's say that F has laid down two sticks of equal length l along the z-axis



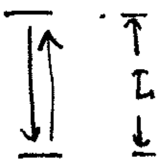
Then F will conclude that it takes time $\Delta t = \frac{l}{c}$ for the light to come to the end of each stick, and thus that the two events of the light hitting the two ends are simultaneous.

At the same time, the origin of F''s coordinate system is moving to the right. So F' will see



so the packet of light moving to the right travels a shorter distance and thus hits its end earlier than the other packet hits the end on the left.

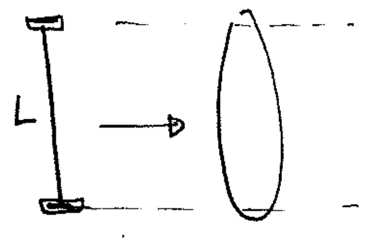
F' and F actually disagree on space and time in several odd ways. First, F believes that F''s clocks run slow. This is easy to see for a model clock which uses light-signals: Place two mirrors perpendicular to the direction of motion of F', and bounce light signals between them



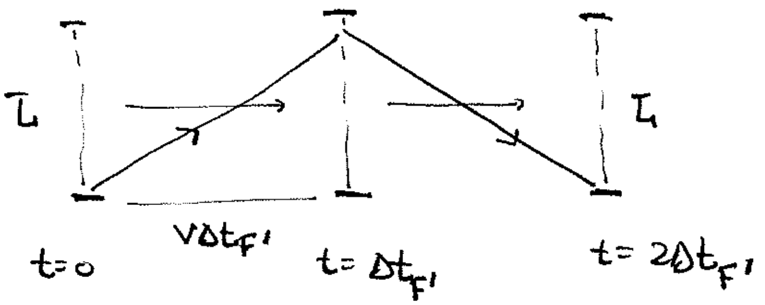
if the mirrors are a distance L apart, the light takes a time

$$2\Delta = 2\frac{L}{c}$$

to go back and forth. F and F' agree on the length L, e.g. F can make a hole of diameter L and see that a clock at rest with respect to F' just fits through



But F thinks that each tick of F''s clock takes longer than each tick of his clock, since in one tick of F''s clock the light must travel:



a distance $[L^2 + (v\Delta t_{F'})^2]^{1/2}$. Since light travels at c, F sees this take a time

$$\Delta t_{F'} = \frac{[L^2 + v^2(\Delta t_{F'})^2]^{1/2}}{c}$$

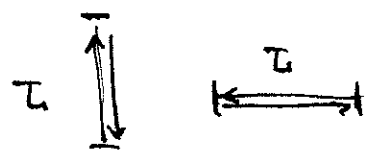
so that

$$(\Delta t_{F'})^2\left(1 - \frac{v^2}{c^2}\right) = \frac{L^2}{c^2}$$

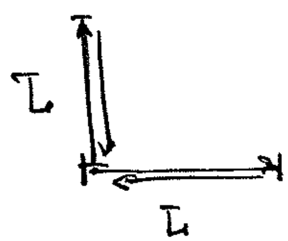
$$\Delta t_{F'} = \frac{1}{\sqrt{1 - v^2/c^2}}\,\Delta$$

F thinks that F$'$'s clock runs slow by a factor $\frac{1}{\sqrt{1-v^2/c^2}}$. By the same analysis, F$'$ thinks the same thing about F's clocks. We have done this analysis with a very special kind of clock but, if the laws of physics are the same in each inertial frame, any other set of clocks based on mechanical principles will have to give the same conclusion.

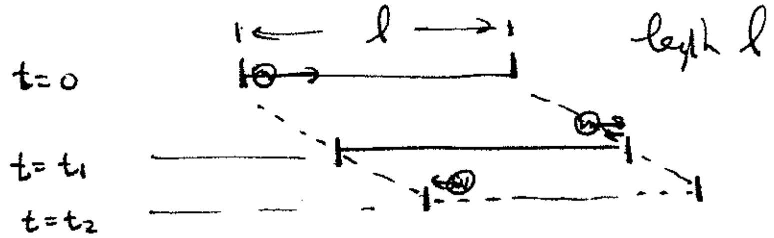
Let's apply this conclusion to clocks made of light and mirrors, now oriented along the z direction. F$'$ sees for his clocks



For each clock, the tick takes $2\Delta = 2\frac{L_1}{c}$. We could position the clocks in a way that it is clear that they tick at the same rate:



Now analyze this from the viewpoint of F. According to F, the tick takes a time $2\Delta t_{F'}$, during which the end of the clock moves by $2V\,\Delta t_{F'}$. F will see:



with $t_1 + t_2 = 2\,\Delta t_{F'}$

$$t_1 = \frac{l + v t_1}{c} \qquad t_2 = \frac{l - V t_2}{c}$$

so
$$t_1 = \frac{l}{c-V} \qquad t_2 = \frac{l}{c+V}$$

$$2\,\Delta t_{F'} = \frac{l}{c-V} + \frac{l}{c+V} = 2\,\frac{l}{c}\,\frac{1}{1-V^2/c^2}$$

Now, this must agree with the previous result

$$2\,\Delta t_{F'} = 2\,\Delta \frac{1}{\sqrt{1-V^2/c^2}}$$

so l, the length of the clock as measured by F, cannot be equal to L_0. It must be

$$l = L_0 \sqrt{1-V^2/c^2}$$

Again, F' must also see F's lengths contracted by $\sqrt{1-V^2/c^2}$.

This is a set of odd conclusions. If F and F' are in relative motion, with relative velocity v.

① Events that F thinks are simultaneous appear to happen earlier for F' if they are ahead in the direction of F's motion and later if they are behind.

② F thinks that F''s clocks run slow by a factor

$$\frac{1}{\sqrt{1-v^2/c^2}}$$ (time dilatation). F' thinks the same about F's clocks.

③ F thinks that F''s lengths are contracted along the direction of motion by a factor $\sqrt{1-v^2/c^2}$ (length contraction or "Lorentz contraction") F' thinks the same about F.

④ At least, F and F' agree on lengths perpendicular to their relative velocity.

These conclusions don't even seem to be self-consistent. But they are. We'll see this, from a better perspective, next time.