The Wave Equation in Cylindrical Coordinates

March 2

In the previous lecture, we thoroughly analyzed the case of waves in a rectangular pipe. Another example that we commonly meet is a pipe with a circular cross-section. Let's now analyze this case.

Begin with the scalar wave equation. This can again be written

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} - \nabla_\perp^2 \right) \phi = 0 \qquad \phi = 0 \text{ on the wall}$$

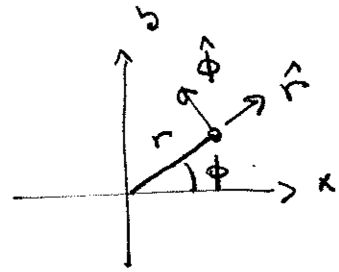
where $$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

But, it will be easier to analyze this system if we convert from rectangular coordinates (x, y, z) to cylindrical coordinates (r, ϕ, z):

$$x = r\cos\phi \qquad y = r\sin\phi$$

Recall that

$$\hat{r} = (\cos\phi, \sin\phi)$$
$$\hat{\phi} = (-\sin\phi, \cos\phi)$$



now
$$\vec{\nabla}_\perp = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}$$

$$\nabla_\perp^2 = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}\right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \underbrace{\hat{r}\cdot\hat{\phi}}_{0}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)\frac{\partial}{\partial \phi}$$

$$+ \hat{\phi}\frac{1}{r}\left(\frac{\partial \hat{r}}{\partial \phi}\right)\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial \phi}\hat{\phi}\right)\frac{\partial}{\partial \phi}$$

now $\frac{\partial}{\partial r}\hat{r} = \frac{\partial}{\partial r}\hat{\phi} = 0 \qquad \hat{r}\cdot\hat{\phi} = 0$

$$\frac{\partial \hat{r}}{\partial \phi} = \hat{\phi} \qquad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{r} \qquad \text{so}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

If we can find eigenfunctions

$$-\nabla_\perp^2 f_{nm}(r,\phi) = \lambda_{nm} f_{nm}(r,\phi)$$

satisfying the boundary condition on a pipe of radius a

$$f_{nm}(a,\phi) = 0$$

Then the wave solutions in the pipe are

$$\phi = \mathrm{Re}\, \phi_0\, e^{-i\omega t + ikz} f_{nm}(r,\phi)$$

with

$$\omega^2 = c^2[k^2 + \lambda_{nm}]$$

the frequencies

$$\omega_{nm} = c\sqrt{\lambda_{nm}}$$

are the cutoff frequencies for the various modes. This eigenvalue problem comes up in many other contexts. For example, the resonant frequencies of a circular drumhead — a 2-dimensional elastic medium constrained a circle of radius $\underline{a}$ are given by

$$\omega_{nm} = c\sqrt{\lambda_{nm}}$$

We can easily get halfway through this eigenvalue problem. Let

$$f_{nm}(r,\phi) = f_{nm}(r)\, e^{im\phi}$$

then $$\frac{\partial^2}{\partial\phi^2} f_{nm} = -m^2 f_{nm}$$

The eigenvalue equation simplifies to

$$\left[-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{m^2}{r^2} \right] f_{nm}(r) = \lambda_{nm} f_{nm}(r)$$

since $\lambda_{nm} > 0$, we can write $\lambda_{nm} = k_{nm}^2$

Now divide through by k_{nm}^2 & let $z = k_{nm} r$. The equation for f becomes

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \left(1 - \frac{m^2}{z^2}\right) \right] f_{nm}(z) = 0$$

or

$$\left[z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 - m^2) \right] f_{nm}(z) = 0$$

with the boundary condition

$$f_{nm}(k_{nm} a) = 0$$

This differential equation is called Bessel's equation. The solutions regular at $z = 0$ (or $r = 0$) are called Bessel functions $J_m(z)$. Bessel functions arise naturally in the eigenvalue problem in cylindrical coordinates, and also in other ~~interesting~~ contexts. So

I would like to digress and study their properties in detail.

First of all, study the properties of Bessel functions near $z=0$. Look for a solution for small z of the form

$$J_m(z) \sim z^\alpha + O(z^{\alpha+1})$$

Plugging this into the differential equation

$$\alpha(\alpha-1)\, z^\alpha + \alpha\, z^\alpha + O(z^{\alpha+2}) - m^2 z^\alpha = 0$$

$$(\alpha^2 - m^2)\, z^\alpha + O(z^{\alpha+2}) = 0$$

so $\qquad \alpha = \pm m$

Since Bessel's equation is a second-order differential equation, it has two linearly independent solutions. One of these can be chosen to behave as z^m near $z=0$. The other behaves as z^{-m}. By convention, the solution which is regular at the origin is called $J_m(z)$; this solution is normalized to

$$J_m(z) \sim \frac{(\frac{1}{2}z)^m}{m!} \quad \text{as } z \to 0$$

The solution that blows up as $z \to 0$ is called the Neumann function

$$Y_m(z) \sim -\frac{1}{\pi}(m-1)!\,(\tfrac{1}{2}z)^{-m} \quad \text{as } z \to 0$$

It is not difficult to construct a power series representation of $J_m(z)$. Rewrite Bessel's equation as

$$\left[z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} - m^2 \right] J_m = - z^2 J_m$$

try a solution

$$J_m(z) = z^m \sum_{k=0}^{\infty} C_k \, z^{2k}$$

The coefficient of z^{m+2k} on each side of the equation is

$$[(m+2k)(m+2k-1) + (m+2k) - m^2] \, C_k = - C_{k-1}$$

$$((m+2k)^2 - m^2) \, C_k = - C_{k-1}$$

$$C_k = - \frac{1}{4k(k+m)} C_{k-1}$$

so if we start from $C_0 = \left(\frac{1}{2}\right)^m \frac{1}{m!}$ $\quad C_1 = -\left(\frac{1}{2}\right)^m \frac{1}{2^2} \frac{1}{1!} \frac{1}{(m+1)!}$

and we find

$$J_m(z) = \left(\tfrac{1}{2} z\right)^m \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4} z^2\right)^k}{k! \, (m+k)!}$$

This series is not very useful for large z. To understand the behavior of J_m at large z, try writing

$$J_m(z) = \frac{1}{\sqrt{z}} \, g(z)$$

$g(z)$ satisfies

$$\left[z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2-m^2) \right] \frac{1}{\sqrt{z}}\, g = 0$$

$$z^2\left[\frac{3}{2}\frac{1}{2}\frac{1}{z^{5/2}}\, g - \cancel{2\,\frac{1}{2}\frac{1}{z^{3/2}}\, g'} + \frac{1}{z^{1/2}}\, g'' \right] + z\left[-\frac{1}{2}\frac{1}{z^{3/2}}\, g + \cancel{\frac{1}{z^{1/2}}\, g'} \right]$$
$$+ (z^2-m^2)\frac{1}{z^{1/2}}\, g = 0$$

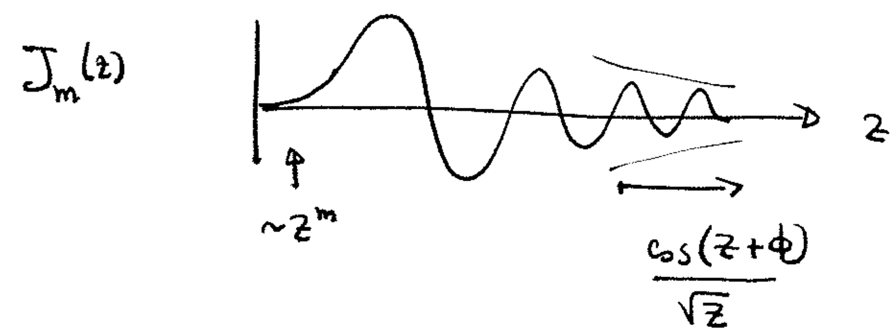
$$z^{3/2}\,(g'' + g) + \frac{1}{z^{1/2}}\left(\frac{3}{4} - \frac{1}{2} - m^2\right) g = 0$$

so that
$$g'' + g = \frac{m^2 - \frac{1}{4}}{z^2}\, g$$

As $z \to \infty$, we can ignore the right-hand side, then
$$g \sim A\cos z + B\sin z$$

so
$$J_m(z) \sim \frac{\cos(z+\phi)}{\sqrt{z}}$$

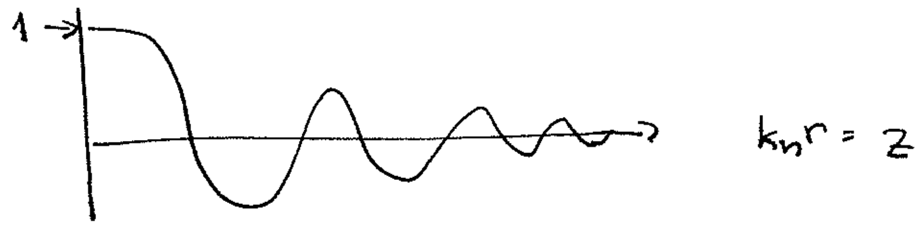
Now we have a fairly convincing qualitative picture of J_m!



I'll make this picture more precise in a moment. But we can already see how to get the eigenfunctions we need on p. 4. Start with $m = 0$. The eigenfunctions are of the form

$$f_{n0}(r,\phi) = J_0(k_{n0} r)$$

where $J_0(k_n r)$ has the form



Now we adjust k_n so that $k_{n0} r$ is a zero of $J_0(z)$. Happily these zeros are tabulated [see eg. Table 9.5 of Abramowicz + Stegun's Handbook of Mathematical Functions.]

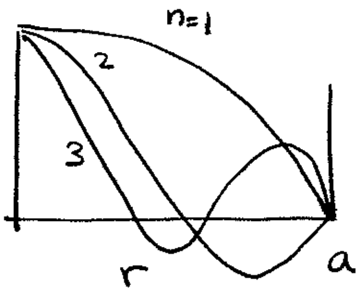
Let z_{nm} be the n^{th} zero of J_m:

$$\begin{array}{lll} z_{10} = 2.4048 & z_{11} = 3.8317 & z_{12} = 5.1356 \\ z_{20} = 5.5201 & z_{21} = 7.0156 & z_{22} = 8.4172 \\ z_{30} = 8.6537 & z_{31} = 10.1735 & z_{32} = 11.6198 \\ \vdots & \vdots & \vdots \end{array}$$

then let $$k_{n0} = \frac{z_{n0}}{a}$$

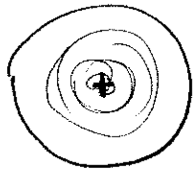
This gives a series of functions

$J_0(k_{n0} r)$

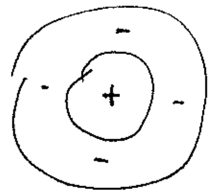


that correspond to the wave profiles

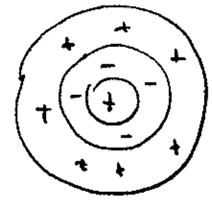
n = 1



n = 2



n = 3



$$\omega_{10} = c\,\frac{z_{10}}{a} \cong 2.4\,\frac{c}{a} \qquad \omega_{20} = c\,\frac{z_{20}}{a} \cong 5.5\,\frac{c}{a} \qquad \omega_{30} = c\,\frac{z_{30}}{a} \cong 8.7\,\frac{c}{a}$$

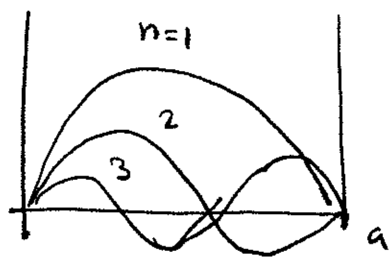
For $m=1$, the eigenfunctions are of the form

$$f_{n1}(r,\phi) = J_1(k_{n1} r)\, e^{i\phi}$$

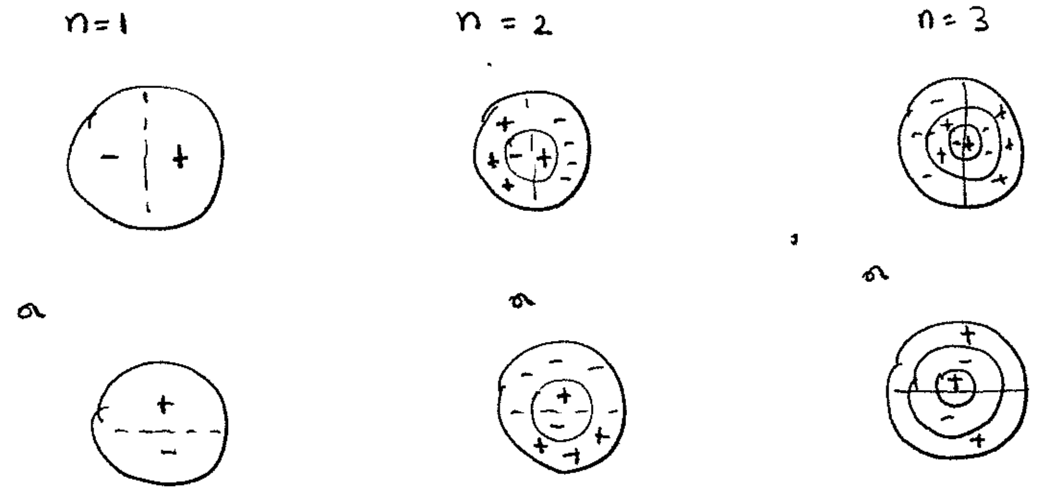
$$= J_1(k_{n1} r)\, \cos(\phi + \phi_0)$$

the radial functions are

$J_1(k_{n1} r)$



the wave profiles are



$$\omega_{11} = c\,\frac{z_{11}}{a} \approx 3.8\,\frac{c}{a} \qquad \omega_{21} = c\,\frac{z_{21}}{a} \cong 7.0\,\frac{c}{a} \qquad \omega_{31} = c\,\frac{z_{31}}{a} \cong 10.2\,\frac{c}{a}$$

similarly

$$f_{nm}(r,\phi) = J_m(k_{nm} r)\, e^{im\phi}$$

with $\quad k_{nm} = \dfrac{z_{nm}}{a} \qquad \omega_{nm} = z_{nm}\dfrac{c}{a}$

Now that we understand how to use the Bessel functions, let's introduce some additional formalism that lets us understand them more precisely. As we did for Legendre polynomials, I'd like to construct the generating function for Bessel functions and use it to analyze their properties.

Consider, then, the function

$$G(z,\theta) = e^{iz\cos\theta}$$

I claim that

$$G(z,\theta) = \sum_{m=-\infty}^{\infty} i^m J_m(z)\, e^{im\theta}$$

Let me first show that the coefficients in this series do indeed satisfy Bessel's equation

$$\left[z^2 \frac{d}{dz^2} + z\frac{d}{dz} + z^2\right] G(z,\theta)$$

$$= \left[-z^2\cos^2\theta + iz\cos\theta + z^2\right] G(z,\theta)$$

$$= \left[z^2\sin^2\theta + iz\cos\theta\right] G(z,\theta)$$

$$= -\frac{\partial^2}{\partial\theta^2} G(z,\theta)$$

so

$$\sum_{m=\infty}^{\infty} \left[z^2\frac{d}{dz^2} + z\frac{d}{dz} + z^2\right] J_m(z)\cdot (i)^m e^{im\theta} = \sum_{m=-\infty}^{\infty} m^2 J_m(z)\,(i)^m e^{im\theta}$$

and, since the functions $e^{im\theta}$ are orthogonal, we are done.

This gives the representation

$$J_m(z) = (-i)^m \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{iz\cos\theta}\, e^{-im\theta}$$

We can check this by expanding

$$e^{iz\cos\theta} = \sum_n \frac{(iz)^n}{n!}(\cos\theta)^n$$

$$= \sum_n \left(\frac{iz}{2}\right)^n \frac{1}{n!}\left(e^{i\theta}+e^{-i\theta}\right)^n$$

Expanding the power gives a series of terms $e^{ik\theta}$, of which only the term $e^{im\theta}$ survives the integral. This term exists only when $n = m + 2k$ $k = 0, 1, 2, \dots$ and in that case

$$\left(e^{i\theta}+e^{-i\theta}\right) \rightarrow \frac{n!}{(m+k)!\; k!}\, e^{im\theta}$$

then we find again

$$J_m(z) = \left(\tfrac{1}{2}z\right)^m \sum_{k=0}^{\infty} \frac{\left(-\tfrac{1}{4}z^2\right)^k}{(m+k)!\; k!} \qquad (m \geq 0)$$

even with the correct normalization. For $m<0$, this definition gives $J_{-m}(z) = (-1)^m J_m(z)$

Now let's use this representation of J_m to study J_m for $z \to \infty$. In this limit, the integral over θ oscillates rapidly except at two special points, $\theta = 0, \pi$, where the function $\cos\theta$ is approximately stationary.

Near $\theta = 0$ we can approximate

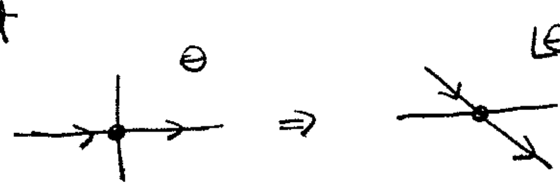
$$\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

so this part of the integral is well approximated by

$$(-i)^m \int \frac{d\theta}{2\pi}\, e^{iz}\, e^{-iz\,\theta^2/2}\, e^{-im\theta}$$

If we regard the ~~integral~~ over θ as a complex plane contour integral and push it over so that

$$\theta = e^{-i\pi/4}\, t$$



then this integral becomes

$$(-i)^m\, e^{-i\pi/4}\, \frac{e^{iz}}{2\pi} \int dt\, e^{-z t^2/2}\, e^{-e^{i\pi/4} m t}$$

this is just a Gaussian integral

$$= e^{-im\pi/2}\, \frac{e^{-i\pi/4}\, e^{iz}}{2\pi} \int dt\, e^{-\frac{z}{2}\left(t + \frac{e^{i\pi/4} m}{2z}\right)^2}\, e^{\frac{1}{2} i \frac{m^2}{z}}$$

$$= e^{-im\pi/2}\, e^{i\pi/4}\, e^{iz}\, \frac{1}{\sqrt{2\pi z}} \left(1 + O\left(\tfrac{1}{z}\right)\right)$$

similarly, the vicinity of $\theta = \pi$ gives

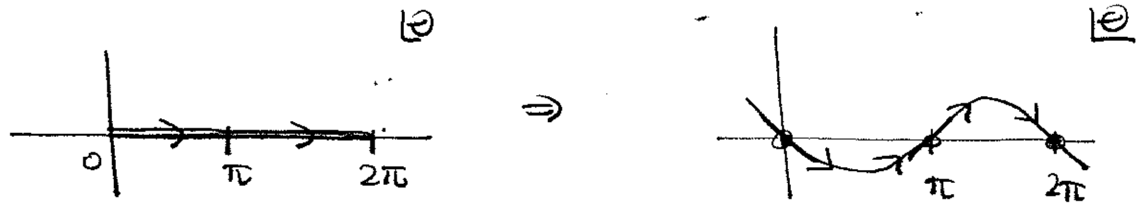
$$= e^{im\pi/2}\, e^{-i\pi/4}\, e^{-iz}\, \frac{1}{\sqrt{2\pi z}}$$

so in all we find

$$J_m(z) \underset{z\to\infty}{\cong} \sqrt{\frac{2}{\pi z}}\ \cos\left(z - m\frac{\pi}{2} - \frac{\pi}{4}\right)\left(1 + \mathcal{O}\left(\tfrac{1}{z}\right)\right)$$

the general form is what we expected, but we also get the phase and normalization.

The deformation of the contour for the $d\theta$ integral is a neat trick called the "method of steepest descents". The final contour is:



On the original contour, the integral involved large cancellations because of the oscillating integrand. On the deformed contour, the integrand is exponentially small except for the peaks at $\theta = 0, \pi$. The value of the integral is independent of which contour we choose, so why not choose cleverly. The method of steepest descents is often very useful in problems where we must evaluate an integral in an asymptotic limit. We'll see it again next term.

Finally, let me just sketch the theory of electromagnetic waves in a circular pipe. As with a rectangular pipe, all modes are either TE or TM. For the TM case, E^z satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} - \nabla_\perp^2 \right) E^z = 0$$

with the Dirichlet boundary condition $E^z = 0$ at $r = a$

Then we get TM_{nm} modes with

$$E^z = \mathrm{Re}\, E_0\, e^{-i\omega t + ikz}\, J_m(k_{nm} r)\, (\cos m\phi \text{ or } \sin m\phi)$$

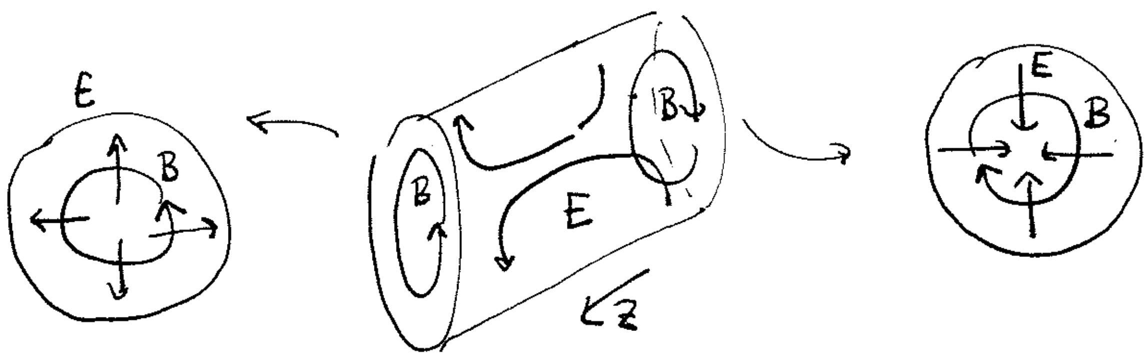
$$\omega_{nm} = z_{nm}\frac{c}{a} \qquad\qquad k_{nm} = \frac{\omega_{nm}}{c} = \frac{z_{nm}}{a}$$

$\vec{E}$, $\vec{B}$ in the transverse direction are given by

$$\vec{E} = i\frac{k}{k_{nm}^2}\vec{\nabla}_\perp E^z$$

$$\vec{B} = i\frac{\omega/c^2}{k_{nm}^2}\hat{z}\times\vec{\nabla}_\perp E^z$$

as on p. 12 of the previous lecture. The simplest mode is $m=0$, $n=1$, which has the waveform:



For the TE case, B^z satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} - \nabla_\perp^2\right) B^z = 0$$

with the Neumann boundary condition $\frac{\partial}{\partial r} B^z = 0$ at $r = a$

Then we get $(TE)_{nm}$ modes with

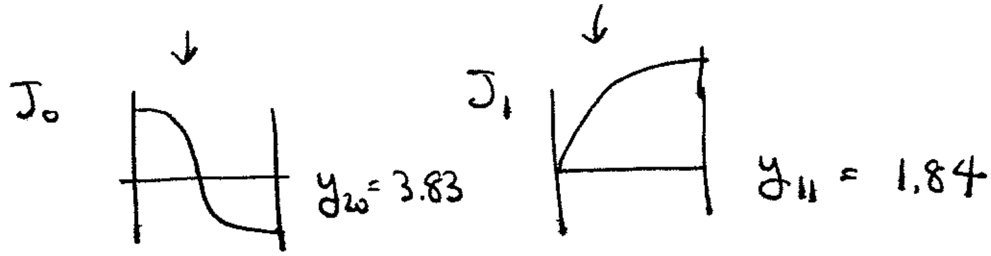
$$B^z = \mathrm{Re}\, B_0\, e^{-i\omega t + ikz} J_m(k_{nm} r)\,(\cos m\phi \text{ or } \sin m\phi)$$

$$\omega_{nm} = y_{nm}\frac{c}{a} \qquad k_{nm} = \frac{\omega_{nm}}{c} = \frac{y_{nm}}{a}$$

where y_{nm} is the n^{th} zero of $\frac{d}{dz} J_m(z)$.

(also tabulated in Abramowitz + Stegun Table 9.5)

$n=1, m=0$ does not give a solution, so the simplest modes are $n=2$ $m=0$ or $n=1$ $m=1$

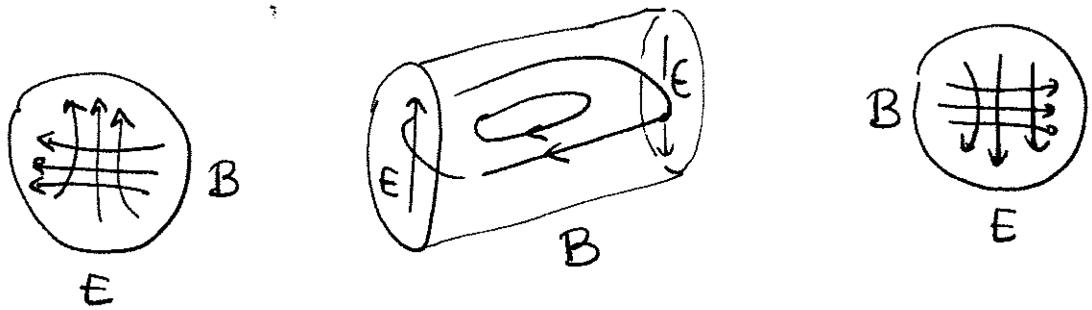


The transverse components of E and B are reconstructed from:

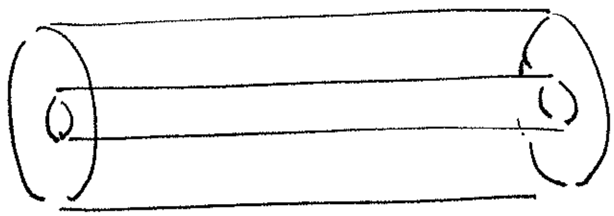
$$\vec{B} = \frac{ik}{k_{nm}^2}\vec{\nabla}_\perp B^z \qquad \vec{E} = \frac{-i\omega}{k_{nm}^2}\,\hat{z}\times\vec{\nabla}_\perp B^z$$

as on p. 10 of the previous lecture.

The very lowest frequency mode turns out to be the $(TE)_{11}$:



As a final example, I would like to discuss one case in which we have a cutoff frequency at $\omega = 0$ and $\vec{E}$ and $\vec{B}$ fields simultaneously transverse. Consider a set of coaxial conducting cylinders of radii a and b



Let $$\vec{E} = \mathrm{Re}\, E_0 \frac{\hat{r}}{r} e^{-i\omega t + ikz}$$

This satisfies $\vec{\nabla} \cdot \vec{E} = 0$ (away from $r=0$) and $E_{\parallel} = 0$ on the walls. Also

$$\vec{\nabla} \times \left(\frac{\hat{r}}{r}\right) = 0$$

since $\frac{\hat{r}}{r}$ is an electrostatic field, so

$$\vec{\nabla} \times \vec{E} = \mathrm{Re}\ E_0\ ik\hat{z} \times \frac{\hat{r}}{r}\ e^{-i\omega t + ikz}$$

$$= \mathrm{Re}\ E_0\ ik\ \frac{\hat{\phi}}{r}\ e^{-i\omega t + ikz}$$

so if $\frac{\partial B}{\partial t} = -\vec{\nabla} \times \vec{E}$, then the correspondg $\vec{B}$ field is

$$\vec{B} = \mathrm{Re}\ \frac{E_0}{\omega} k\ \frac{\hat{\phi}}{r}\ e^{-i\omega t + ikz}$$

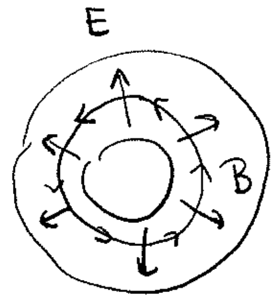
and the whole soln is consistent if

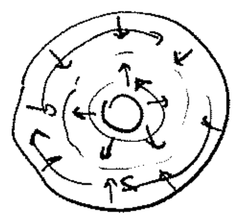
$$\omega = ck$$

$$\vec{E} = \mathrm{Re}\ E_0\ \frac{\hat{r}}{r}\ e^{-i\omega t + ikz}$$

$$\vec{B} = \mathrm{Re}\ \frac{E_0}{c}\ \frac{\hat{\phi}}{r}\ e^{-i\omega t + ikz}$$

in cross section





Higher modes are possible eg.

but these have $\omega_{cutoff} > 0$

and bring in Bessel functions again.