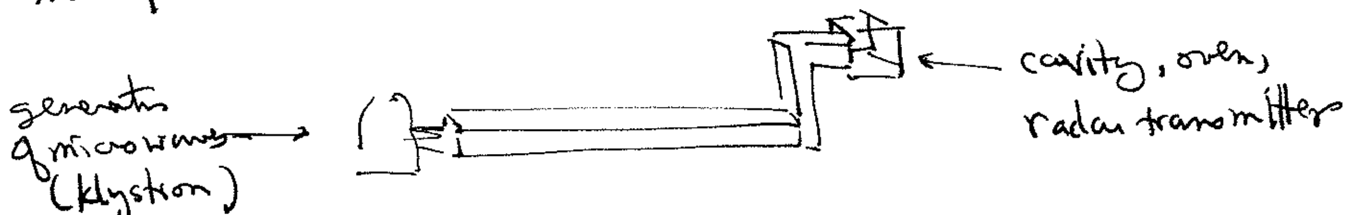
Waveguides

Feb. 26

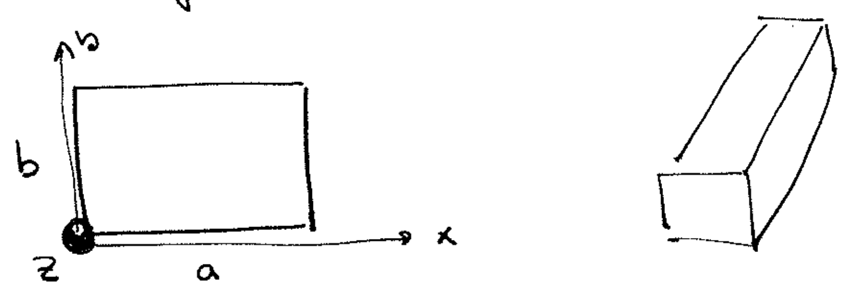
So far, we have discussed electromagnetic waves in unbounded space. In many circumstances, though, it is useful to make a channel for electromagnetic radiation. For microwaves ($\lambda \sim 1\,cm$), we can use a piece of metal pipe to transport radiation from one place to another:



Today light is transported in optical fibers. I'd like to study the properties of electromagnetic waves in such elongated structures.

Begin by using the scalar wave equation as a model.

Consider a pipe with rectangular cross-section



with the condition that $\phi(t,\vec{x})$ obeys the wave equation inside the pipe and

$$\phi(t,\vec{x}) = 0 \quad \text{on the walls.}$$

What are the solutions of the wave equation with these boundary conditions?

We can solve this problem by using the general theory of the wave equation that we developed earlier in the course. Write the wave equation as:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} - \nabla_\perp^2\right)\phi = 0$$

where

$$-\nabla_\perp^2 = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

is the Laplace operator in 2-dimensions. We have this operator together with the Dirichlet boundary condition $\phi = 0$. This gives a Sturm Liouville problem — a well-defined eigenvalue problem for $(-\nabla_\perp^2)$. The solutions, we know, are sin's and cosines — actually only sin's because of the condition that $\phi = 0$ at $x=0$ and $y=0$. So, the eigenfunctions are

$$f_{nm}(x,y) = \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m y}{b}\right)$$

and

$$-\nabla_\perp^2 f_{nm} = \left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right] f_{nm}$$

Now we can look for a solution to the wave equation of the form:

$$\phi(t,\vec{x}) = Re\left[\phi_0 \, e^{ikz - i\omega t} f_{nm}(x,y)\right]$$

applying the wave operator, we find:

$$\left[-\frac{\omega^2}{c^2} + k^2 + \left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right] = 0$$

so
$$\omega^2 = c^2\left[k^2 + \left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right]$$

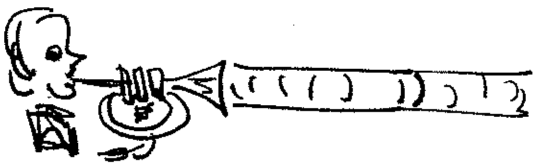
Notice ~~that~~ each wave shape in the transverse plane propagates with a definite minimum frequency

$$\omega_{nm} = c\left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right]^{\frac{1}{2}}$$

"cutoff frequency for the nm mode"

so that
$$\omega^2 = c^2 k^2 + \omega_{nm}^2$$

If we try to force waves into the pipe with profile f_{nm} at a frequency less than ω_{nm}



Then we find that k is pure imaginary

$$k = i\left[\frac{\omega_{nm}^2 - \omega^2}{c^2}\right]^{\frac{1}{2}}$$

and the signal decays exponentially in the channel

$$e^{ikz} = e^{-\frac{1}{c}[\omega_{nm}^2-\omega^2]^{\frac{1}{2}} z}$$

Since $\vec{j}_E = -\frac{\partial\phi}{\partial t}\vec{\nabla}\phi$ and $\frac{\partial\phi}{\partial t}$ and $\hat{z}\cdot\vec{\nabla}\phi$ are 90° out of phase in this situation

$$\hat{z}\langle\vec{j}_E\rangle = \hat{z}\cdot\langle-\frac{\partial\phi}{\partial t}\vec{\nabla}\phi\rangle = 0$$

and no energy is transported down the pipe. In particular the lowest of these basic frequencies is

$$\omega_{11} = c\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^{\frac{1}{2}}$$

The pipe will not propagate any frequency lower than ω_{11}.

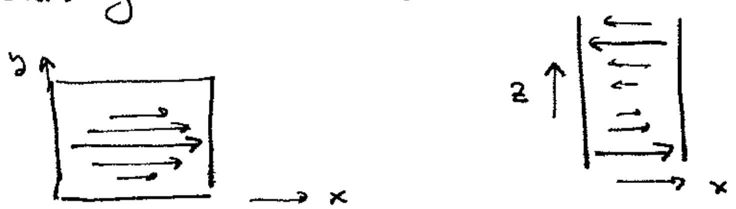
Formally, the wave with f_{nm} is a linear combination of plane waves

$$e^{-i\omega t}\, e^{ikz}\, e^{\pm i\frac{\pi n}{a}x}\, e^{\pm i\frac{\pi m}{b}y}$$

You can think of the guided wave as one of these waves multiply reflecting from the boundaries.

Now generalize this formalism to electromagnetism. For simplicity, I'll consider the container to be a perfect conductor. Then $\vec{E}_{\parallel} = B_{\perp} = 0$ on the walls.

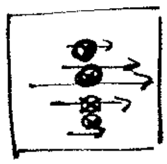
Let's try to imagine what the E and B field configurations in a pipe would look like. An E field that satisfies the boundary conditions is, e.g.



$$\vec{E}(t,x) = \mathrm{Re}\; E_0\,\hat{x}\,\sin\frac{\pi y}{b}\; e^{-i\omega t}\, e^{ikz}$$

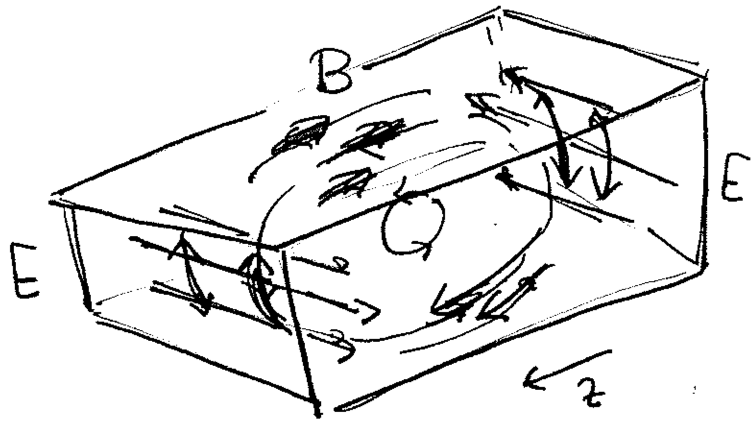
This $\vec{E}$ field has a nonzero curl, which means that a B field should turn on: In particular

$(\vec{\nabla}\times\vec{E})^z$:



$\Rightarrow \dfrac{\partial B^z}{\partial t} \neq 0$

So we get a field pattern like:

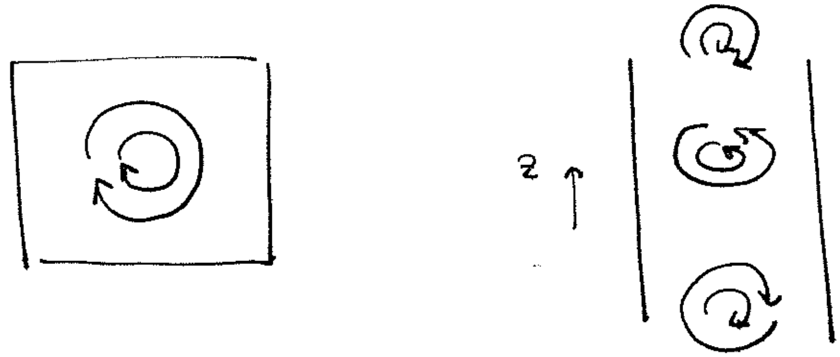


Notice that

$\vec{\nabla}\cdot\vec{B} = 0$ implies that lines of B do not end

$S = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ has a nonzero $\langle\,\rangle$ pointing in the $\hat{z}$ direction

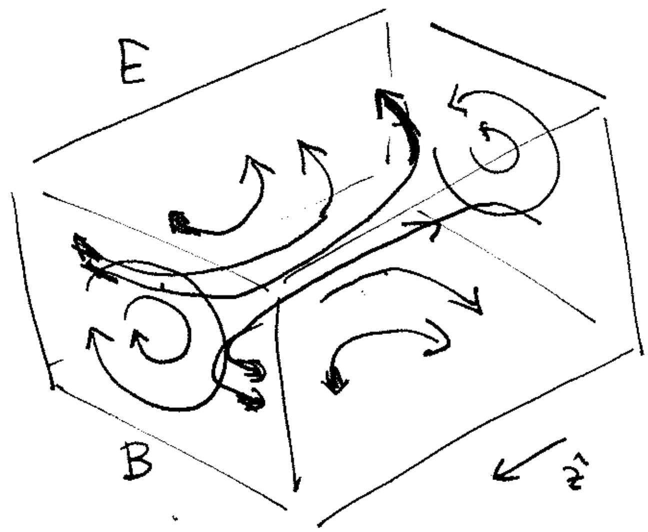
This waveform has $E_z = 0$ but $B_z \neq 0$. Can we construct a waveform with $B_z = 0$? Since $\vec{B}$ has no sources, the field must look like



This field configuration obviously has a curl, and

$$(\vec{\nabla} \times \vec{B})_z \neq 0 \Rightarrow \frac{\partial E^z}{\partial t} \neq 0$$

So we get a field pattern like



Again, $\vec{B}$ is sourceless, $\vec{E}$ ends on the walls,

$$\langle \vec{S} \rangle = \langle \frac{1}{\mu_0} \vec{E} \times \vec{B} \rangle$$ is nonzero and parallel to $\hat{z}$

These two pattern are called, respectively,

TE (transverse electric $\rightarrow$ $E^z = 0$)

TM (transverse magnetic $\rightarrow$ $B^z = 0$)

waves. It is not difficult to show that it is impossible in this geometry for both E and B to be transverse:

$$E^z = 0 \quad \Rightarrow \quad (\text{using } \vec{\nabla}\cdot\vec{E} = 0) \qquad \frac{\partial}{\partial x} E^x + \frac{\partial}{\partial y} E^y = 0$$

$$B^z = 0 \quad \Rightarrow \quad \text{using } \frac{\partial}{\partial t}\vec{B} = -\vec{\nabla}\times\vec{E} \qquad \frac{\partial}{\partial x} E^y - \frac{\partial}{\partial y} E^x = 0$$

The second equation implies that $(E^x, E^y) = \left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right)\phi$ for some potential ϕ. The first equation implies that

$$-\nabla_\perp^2 \phi = 0$$

But $\phi = \text{const}$ on the boundary, so the unique solution is $\phi = \text{const}$ everywhere, $E^x = E^y = 0$.

If this is so, we ought to be able to construct the explicit waveforms by the following strategy: Write a formula for $E^z(t,\vec{x})$ and $B^z(t,\vec{x})$, then use Maxwell's equations to deduce E^x E^y B^x, B^y. For simplicity, we can consider separately the cases.

TE : $\quad E^z = 0 \quad B^z \neq 0$

TM : $\quad E^z \neq 0 \quad B^z = 0$

Consider first TE:

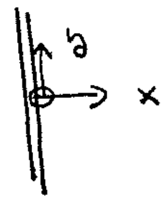
B^z satisfies the wave equation inside the pipe

$$\left(\frac{1}{c^2} \frac{\partial}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \nabla_\perp^2 \right) B^z = 0$$

To find the boundary conditions on B^z, consider the equation

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}$$

evaluated on a wall $\perp$ to $\hat{x}$



Taking $\hat{y} \cdot \left(\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} \right) = 0$

we find

$$\frac{\partial}{\partial z} B^x - \frac{\partial}{\partial x} B^z - \frac{1}{c^2} \frac{\partial}{\partial t} E^y = 0$$

Now, the boundary conditions

$$E_\parallel = 0 \quad \Rightarrow \quad E^y = 0$$

$$B_\perp = 0 \quad \Rightarrow \quad B^x = 0$$

so

$$\frac{\partial}{\partial x} B^z = 0 \text{ , more generally}$$

$$\hat{n} \cdot \vec{\nabla} B^z = 0 \qquad \text{(Neumann b.c.'s)}$$

so, let's write the most general wave B^z consistent with these boundary conditions, and then construct E^x E^y B^x B^y.

For B^z, we want a form

$$B^z = \mathrm{Re}\, B_0\, e^{ikz - i\omega t}\, g(x,y)$$

where $g(x,y)$ is an eigenfunction of $-\nabla_\perp^2$ with Neumann boundary conditions. These eigenfunctions are

$$g_{nm}(x,y) = \cos\frac{\pi n x}{a}\cos\frac{\pi m y}{b}$$

The wave with the choice g_{nm} is called the TE_{nm} mode. This mode has a cutoff frequency

$$\omega_{nm}^{(TE)} = c\left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right]^{1/2}; \quad k_{nm}^{(TE)} = \frac{\omega_{nm}^{(TE)}}{c}$$

and

$$\omega^2 = c^2 k^2 + \omega_{nm}^{(TE)\,2}$$

we will see ~~that~~ $n=m=0$ does not yield a sensible mode, so the lowest cutoff frequency is

$$\omega_{10}^{(TE)} = c\frac{\pi}{a} \qquad \text{or} \qquad \omega_{01}^{(TE)} = c\frac{\pi}{b}$$

Now construct the corresponding $E^x\ E^y\ B^x\ B^y$:

$$\left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times\vec{B}\right) \xrightarrow{\cdot\hat{x}} \frac{1}{c^2}\frac{\partial E^x}{\partial t} = \frac{\partial}{\partial y}B^z - \frac{\partial}{\partial z}B^y$$

$$\left(\frac{\partial B}{\partial t} = -\vec{\nabla}\times\vec{E}\right) \xrightarrow{\cdot\hat{y}} \frac{\partial B^y}{\partial t} = -\frac{\partial}{\partial z}E^x + \frac{\partial}{\partial x}E^z$$

now use $E^z = 0 \quad \frac{\partial}{\partial t}\to -i\omega \quad \frac{\partial}{\partial z}\to ik$

$$ik \qquad -\frac{i\omega}{c^2} E^x + ik B^y = \frac{\partial}{\partial y} B^z$$

$$+i\frac{\omega}{c^2} \qquad -i\omega B^y + ik E^x = 0$$

so

$$E^x = \frac{1}{\omega^2/c^2 - k^2} \, i\omega \frac{\partial}{\partial y} B^z$$

$$B^y = \frac{1}{\omega^2/c^2 - k^2} \, ik \frac{\partial}{\partial y} B^z$$

now $\omega^2/c^2 - k^2 = k_{nm}^{(TE)}$ so, (Note that, for $n=m=0$, $k_{nm}=0$ and the solution breaks down)

$$E^x = i\frac{\omega}{k_{nm}^2}\frac{\partial}{\partial y} B^z \qquad B^y = i\frac{k}{k_{nm}^2}\frac{\partial}{\partial y} B^z$$

similarly

$$E^y = -i\frac{\omega}{k_{nm}^2}\frac{\partial}{\partial x} B^z \qquad B^x = i\frac{k}{k_{nm}^2}\frac{\partial}{\partial x} B^z$$

putting $B^z = \mathrm{Re}\; e^{ikz} e^{-i\omega t} \cos\frac{\pi y}{b}$ $\quad (TE_{01})$

we find:

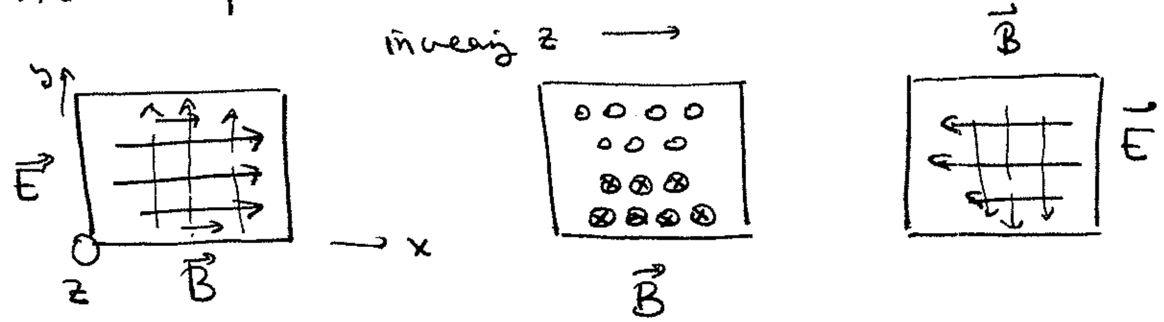
$$(E^x, E^y) = i\frac{\omega}{k_{nm}^2}\left(-\frac{\pi}{b}\sin\frac{\pi y}{b},\ 0\right)$$

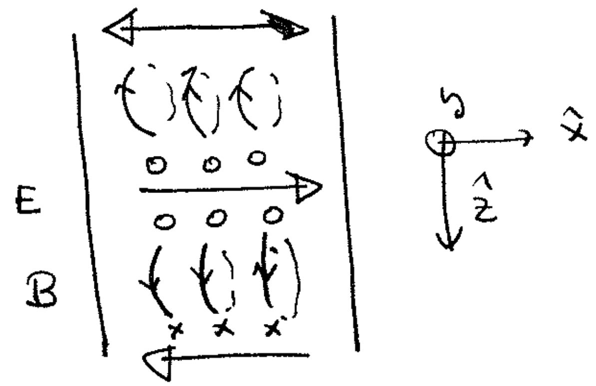
$$(B^x, B^y) = i\frac{k}{k_{nm}^2}\left(0,\ -\frac{\pi}{b}\sin\frac{\pi y}{b}\right)$$

Notice that E^x, B^y are 90° out of phase with B^z, but they are in phase with each other, so that

$$\langle S^z \rangle = \langle \frac{1}{\mu_0}(\vec{E}\times\vec{B})^z \rangle > 0$$

The wave pattern in the pipe is





which is just what is sketched on p. 5

The theory of the TM modes is similar. Now we set $B^z = 0$, $E^z \neq 0$. E^z is a component parallel to the boundary so $E^z|_{\text{wall}} = 0$, a Dirichlet boundary condition. Then E^z solves exactly the scalar wave equation studied at the beginning of this lecture. The solution, again, is

$$E^z = \mathrm{Re}\; E_0\, e^{ikz - i\omega t} f_{nm}(x,y)$$

where

$$f_{nm} = \sin\frac{\pi n}{a}x \quad \sin\frac{\pi m}{b}$$

and the cutoff frequency is

$$\omega_{nm}^{(TM)} = c\left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right]^{\frac{1}{2}} \qquad k_{nm}^{(TM)} = \frac{\omega_{nm}^{(TM)}}{c}$$

This formula is the same as on p. 9, but here

$$n, m \geq 1$$

The lowest cutoff frequency is $\omega_{11}^{TM} = c\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^{\frac{1}{2}}$.

Formula for $E^x\ E^y\ B^x B^y$ can be worked out as before, from the $\nabla\times E$, $\nabla\times B$ Maxwell equations. The results are:

$$E^x = i\,\frac{k}{k_{nm}^2}\,\frac{\partial}{\partial x}E^z \qquad E^y = \frac{ik}{k_{nm}^2}\,\frac{\partial}{\partial y}E^z$$

$$B^x = \frac{-i\omega/c^2}{k_{nm}^2}\,\frac{\partial}{\partial y}E^z \qquad B^y = \frac{i\omega/c^2}{k_{nm}^2}\,\frac{\partial}{\partial x}E^z$$

For $n = m = 1$, we find

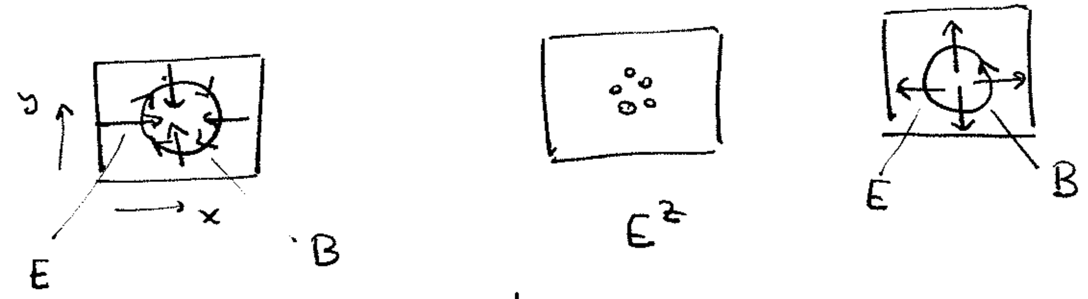
$$(E^x, E^y) \sim \left(\frac{\pi}{a}\cos\frac{\pi}{a}x\,\sin\frac{\pi}{b}y,\ \frac{\pi}{b}\sin\frac{\pi}{a}x\,\cos\frac{\pi}{b}y\right)$$

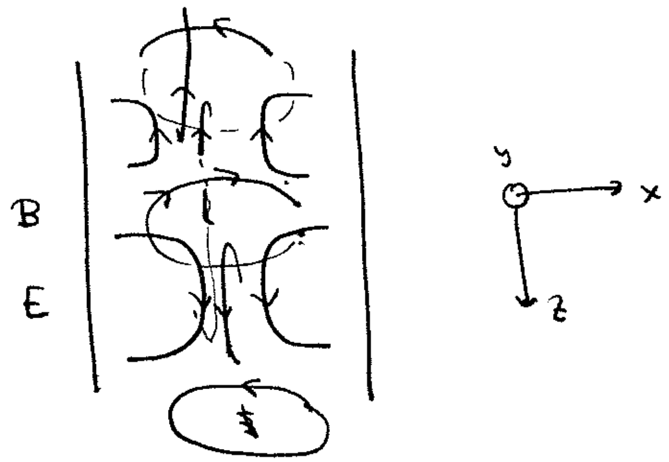
$$(B^x, B^y) \sim \left(-\frac{\pi}{b}\sin\frac{\pi}{a}x\cos\frac{\pi}{b}y,\ \frac{\pi}{a}\cos\frac{\pi}{a}x\,\sin\frac{\pi}{b}y\right)$$

With these components 90° out of phase with E^z.

The wave pattern in the pipe is

moving $z \longrightarrow$





which is (a somewhat better version of) what is sketched on p.6.

If we have a travelling wave solution in a pipe, we can also create a standing wave solution. Let's do this first for the scalar wave equation. If

$$\phi_{nm} = \mathrm{Re}\, \Phi_0\, e^{ikz - i\omega t} f_{nm}(x,y)$$

is a solution to the wave equation, so is

$$\tilde{\Phi}_{nmk} = \mathrm{Re}\, \frac{\Phi_0}{2}\left[e^{ikz-i\omega t} f_{nm}(x,y) + e^{-ikz} e^{-i\omega t} f_{nm}(x,y)\right]$$

(a superposition of two waves going in opposite directions down the pipe) More explicitly:

$$\phi_{nmk} = \phi_0 \cos kz \sin \frac{\pi n x}{a} \sin \frac{\pi m}{b} y \cos \omega t$$

This solution has zeros in z. If we shift the origin of z, we can find a solution which satisfies the boundary condition $\phi = 0$ at

$$x = (0, a) \qquad y = (0, b) \quad \underline{\text{and}} \quad z = (0, c)$$

$$\phi_{nml} = \phi_0 \sin \frac{\pi n}{a} x \sin \frac{\pi m}{b} y \sin \frac{\pi l}{c} z$$

$$\cdot \cos \omega t$$

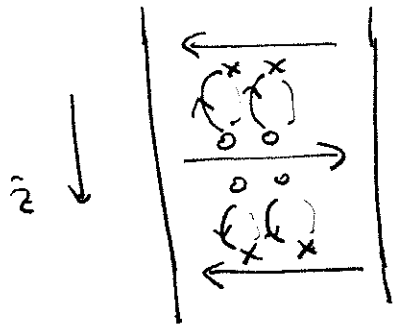
with $$\omega = c \left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2 + \left(\frac{\pi l}{c}\right)^2 \right]^{1/2}$$

These are the oscillating solutions of the wave equation in a finite box. The frequencies are <u>discrete</u>. In fact they are exactly

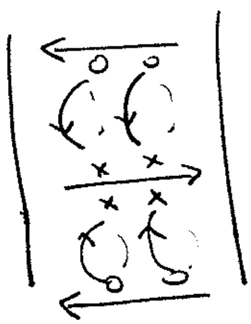
$$\omega_a = c\,(\lambda_a)^{1/2}$$

where λ_a are the eigenvalues of $(-\nabla^2)$ in 3-dimensions with the Dirichlet boundary condition $\phi = 0$.

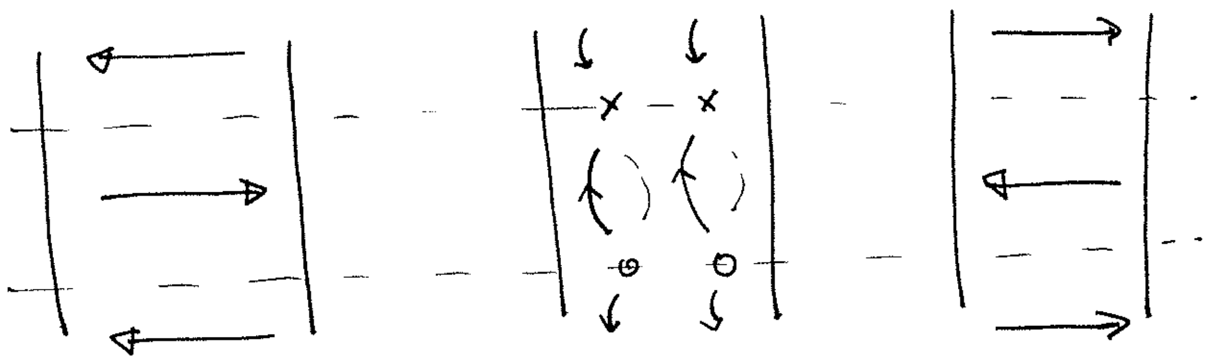
The oscillating solutions of Maxwell's equations in a conducting box can be found in the same way. Start with e.g. a TE solution propagating in the $\hat{z}$ direction



Superpose a TE solution with the same ω, but propagating in the opposite direction

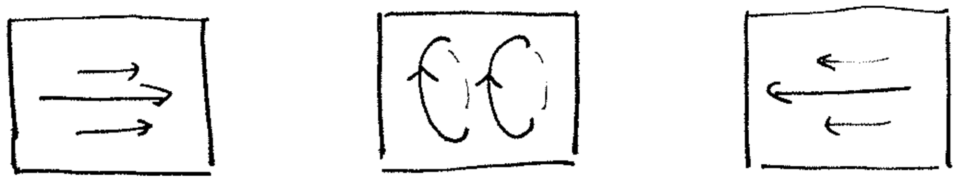


This gives a standing wave solution:



time $\longrightarrow$

Now put conducting walls along the lines indicated by
- - - - : This gives an oscillating field



$t \longrightarrow$

that satisfies the correct boundary conditions

$$E_{\parallel} = 0 \qquad B_z = 0$$

on all 6 walls. If the size in z is c, the frequency is

$$\omega = c\left[\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2\right]^{\frac{1}{2}}$$

In a similar way, we can construct modes of the electromagnetic field in a rectangular cavity of size (a, b, c) which oscillate with frequencies.

$$\omega_{nml} = c\left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2 + \left(\frac{\pi l}{c}\right)^2\right]^{\frac{1}{2}}$$

where at least two of n, m, l must be nonzero.