Feb. 23

Waves in Less Ideal Media - 2

In the previous lecture, we considered waves in a conducting medium using the relation

$$\vec{J} = \sigma \vec{E}$$

This means that the electrons in the material have an average velocity proportional to $\vec{E}$. That is, they are not accelerating but rather have reached terminal velocity. This is a reasonable way to model a dense medium. However, some media of interest are dilute, and it is important to take account of how the electrons in them are actually accelerated and decelerated by the electric forces in an electromagnetic wave. The two most familiar examples of such dilute media are ionized gases and semiconductors.

As a first exercise, consider the medium to be made of electrons and ions, so that it is overall neutral. Treat the ions as static, and treat the electrons as free particles. Let $\vec{X}(t,\vec{x})$ be the displacement of an electron from a nearby ion. Then the medium has a polarization

$$\vec{P} = -e\rho\vec{X}$$

where $(-e)$ is the electron charge and ρ is the density of electrons (#/m³). I will assume that electrons typically move distances much less than λ, the wavelength of an electromagnetic wave. This is a good approximation if the electrons move nonrelativistically ($v \ll c$) while the wave moves at $v \sim c$. Then we can write

$$m\ddot{\vec{X}} = -e\vec{E}(t,\vec{x})$$

$$\ddot{\vec{P}} = \frac{e^2 \rho}{m}\vec{E}(t,\vec{x})$$

Let's put this behavior into Maxwell's equations:

$$\vec{\nabla}\cdot E = -\frac{1}{\epsilon_0}\vec{\nabla}\cdot\vec{P} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} + \mu_0\frac{\partial\vec{P}}{\partial t}$$

Now look for solutions to these equations of the form

$$\vec{E}(t,\vec{x}) = \mathrm{Re}\ \vec{E}_0\, e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

$$\vec{B}(t,\vec{x}) = \mathrm{Re}\ \vec{B}_0\, e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

$$\vec{P}(t,\vec{x}) = \mathrm{Re}\ \vec{P}_0\, e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

The equation of motion of $\vec{P}$ implies

$$-\omega^2\vec{P}_0 = \frac{e^2\rho}{m}\vec{E}_0$$

then

$$\epsilon_0 \vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} = 0 \quad \Rightarrow \quad i\vec{k}\cdot\vec{E}_0\left(\epsilon_0 - \frac{e^2\rho}{m\omega^2}\right)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad i\vec{k}\cdot\vec{B}_0$$

so except at $\omega^2 = \dfrac{e^2\rho}{\epsilon_0 m}$ exactly, $\vec{k}\cdot\vec{E}_0 = \vec{k}\cdot\vec{B}_0 = 0$

$$-\frac{\partial B}{\partial t} = \vec{\nabla}\times\vec{E} \quad \Rightarrow \quad \vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0 \quad \text{as usual}$$

the final equation is

$$i\vec{k}\times\vec{B}_0 = (\epsilon_0\mu_0)\left[-i\omega\vec{E}_0 - \frac{(-i\omega)}{\epsilon_0}\frac{e^2\rho}{m\omega^2}\vec{E}_0\right]$$

now $\vec{k}\times\vec{B}_0 = \vec{k}\times\left(\dfrac{\vec{k}}{\omega}\times\vec{E}_0\right) = -\dfrac{k^2}{\omega}\vec{E}_0$ if $\vec{k}\cdot\vec{E}_0 = 0$

so

$$k^2 = \frac{1}{c^2}\,\omega^2\left(1 - \frac{e^2\rho}{\epsilon_0 m\omega^2}\right)$$

This is a relation:

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}$$

or $\omega^2 = c^2k^2 + \omega_p^2$

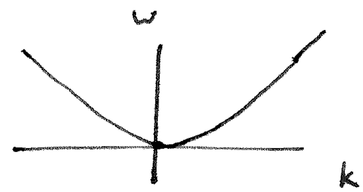
where $\omega_p^2 = \dfrac{e^2\rho}{\epsilon_0 m}$

The relation between ω and k is called the <u>dispersion relation</u>. Recall that

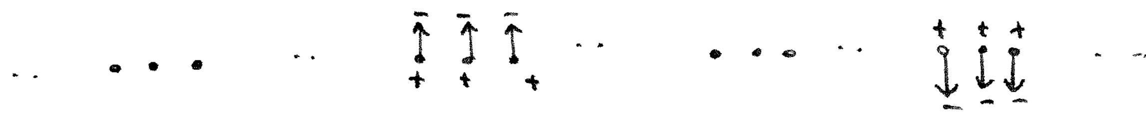
$$\frac{d\omega}{dk}$$

is the group velocity, so wave packets travel slowly when $d\omega/dk$ is small. In this case, the relation between ω and k is

$$\omega = \sqrt{c^2k^2+\omega_p^2}$$



ω_p is called the "plasma frequency". At $\vec{k}=0$, the oscillation is the oscillation of charge "plasma oscillation"

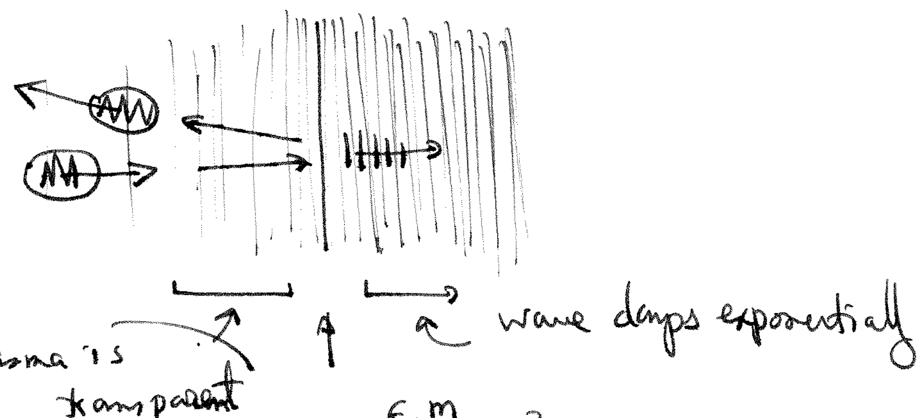


For $ck \gg \omega_p$, the wave becomes an ordinary electromagnetic wave moving at speed c.

Notice that ω_p increases with ρ. So if we go from a very dilute ionized gas to a less dilute one, a wave that begins with $\omega > \omega_p$ can come to a place where $\omega < \omega_p$. If $\omega < \omega_p$,

$$k = \frac{i}{c}\left[\omega_p^2-\omega^2\right]^{1/2}$$

and the wave is damped exponentially in the forward direction. In this case, the wave is reflected:



$$\rho = \frac{\epsilon_0 m}{e^2}\,\omega^2$$

This is the reason that radio waves reflect from the ionosphere.

Now add damping or friction to our equation for $\underline{X}$:

$$m\ddot{\underline{X}} + m\gamma\dot{\underline{X}} = -e\,\vec{E}(t,\vec{x})$$

$$\ddot{\vec{P}} + \gamma\dot{\vec{P}} = \frac{e^2\rho}{m}\,\vec{E}$$

$$\vec{P}_0 = \frac{1}{(-\omega^2 - i\omega\gamma)}\,\frac{e^2\rho}{m}\,\vec{E}_0$$

Then $\vec{j} = \dfrac{\partial\vec{P}}{\partial t}$ is described by

$$\vec{j}_0 = \left(\frac{-i\omega}{-i\omega\gamma - \omega^2}\right)\frac{e^2\rho}{m}\,\vec{E}_0$$

If the damping is large, we have

$$\vec{j}_0 = \frac{e^2\rho}{\gamma m}\vec{E}_0$$

which is a relation of the form $\vec{j} = \sigma\vec{E}$ considered in the previous lecture.

Taking the full form of $\vec{P}$ given on p. 5, the last Maxwell equation takes the form

$$i\vec{k}\times\vec{B}_0 = (\epsilon_0\mu_0)(-i\omega)\vec{E}_0\left(1 + \frac{1}{(-\omega^2 - i\omega\gamma)}\frac{e^2\rho}{\epsilon_0 m}\right)$$

or

$$c^2k^2 = \omega^2\left(1 - \frac{1}{\omega^2 + i\omega\gamma}\frac{e^2\rho}{\epsilon_0 m}\right)$$

$$c^2k^2 = \omega^2 - \frac{\omega}{\omega + i\gamma}\left(\frac{e^2\rho}{\epsilon_0 m}\right)$$

For small γ and $k = 0$,

$$0 = \omega^2 - \omega_p^2\left(1 - \frac{i\gamma}{\omega}\right)$$

so

$$\omega \cong \omega_p - \frac{i\gamma}{2\omega_p}$$

and the behavior $e^{-i\omega t}$ is a slow damping in time:

$$e^{-i\omega_p t}\, e^{-\frac{\gamma}{2\omega_p}t}$$

For fixed $\omega < \omega_p$, we have

$$k = \frac{1}{c}\left[\omega^2 - \omega_p^2 \frac{\omega}{\omega + i\gamma}\right]^{\frac{1}{2}}$$

For $\omega \gg \gamma$ = 1/damping time

$$k \approx \frac{i}{c}\left[\omega_p^2 - \omega^2\right]^{\frac{1}{2}}$$

then if $\vec{E}_0$ is real $\quad \vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \approx \frac{i}{c}\left[\omega_p^2 - \omega^2\right]^{\frac{1}{2}} \hat{k} \times \vec{E}_0$

and so E and B are 90° out of phase.

then $\quad \langle \vec{E} \times \vec{B} \rangle \sim \langle \cos \omega t \sin \omega t \rangle = 0$

and so the exponentially decreasing wave carries no energy ~~into~~ the plasma.

However, in the opposite limit $\quad \omega \ll 1/\gamma$

$$k \approx \frac{1}{c}\left[i \frac{\omega_p^2}{\gamma} \omega - \omega^2\right]^{\frac{1}{2}}$$

and for larger γ we move to the situation

$$k \sim \left(\frac{1+i}{\sqrt{2}}\right) \cdot \frac{\omega_p}{c}\left(\frac{\omega}{\gamma}\right)^{\frac{1}{2}}$$

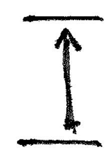
seen in the previous lecture in which k and ω are only 45° out of phase, $\langle \vec{E} \times \vec{B} \rangle \neq 0$, and in fact energy is transported into the plasma and dissipated.

Now add one more complication. Let's consider electrons that are bound to atoms by a harmonic oscillator potential

$$m\left(\ddot{\vec{X}} + \gamma \dot{\vec{X}} + \omega_0^2 \vec{X}\right) = -e \vec{E}(t,\vec{x})$$

The equations to follow actually apply also to the following situation in quantum mechanics: at atom has an energy level separated from the ground state level by energy

$$\Delta E = \hbar\omega_0$$



and the excited level is <u>unstable</u> and radiates back down with lifetime $\tau = 1/\gamma$.

In this situation, we have

$$m\left(-\omega^2 - i\omega\gamma + \omega_0^2\right)\vec{P}_0 = e^2\rho\, \vec{E}_0$$

$$\vec{P}_0 = -\frac{e^2\rho}{m}\frac{1}{(\omega^2+i\omega\gamma-\omega_0^2)}\vec{E}_0$$

Pushing this through the Maxwell equations as before, we find the dispersion relation:

$$c^2k^2 = \omega^2 \left(1 + \frac{e^2\rho}{\epsilon_0 m} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}\right)$$

A settg in which we can understand how this equatn works is that in which we have a dilute gas of molecules with a quantm mechanical transitn at $\Delta E = \hbar\omega_0$.
Then we can take ρ to be small and expand:

$$k \cong \frac{\omega}{c} \left(1 + \frac{e^2\rho}{2\epsilon_0 m} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}\right)$$

Since the index of refractn $n = \frac{ck}{\omega}$, we can write this as

$$n \cong 1 + \frac{e^2\rho}{2\epsilon_0 m} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}$$

$$= 1 + \frac{e^2\rho}{2\epsilon_0 m} \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}$$

Let's explore this formula as ω is increased from small values.
For small ω, n is real and constant

$$n \cong 1 + \frac{e^2\rho}{2\epsilon_0 m} \frac{1}{\omega_0^2}$$

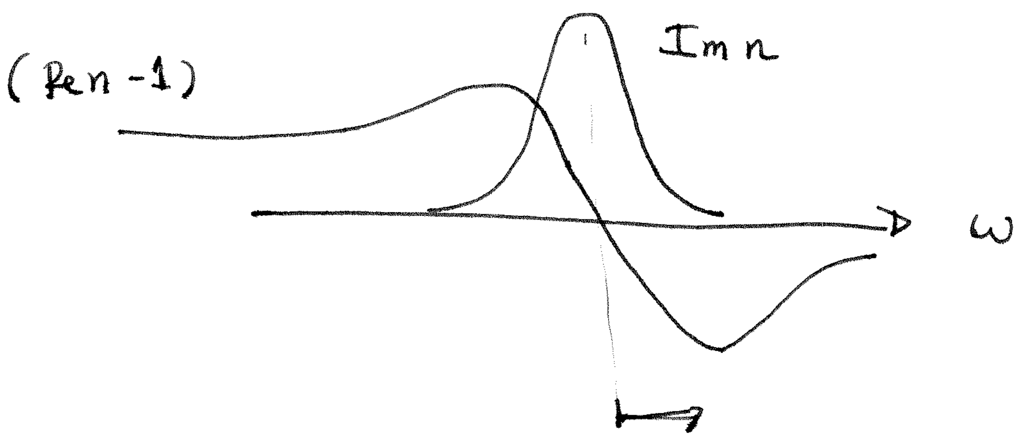
and > 1.

For large ω, $n \to 1$; the wave is oscillating so fast that the atomic vibrations cannot keep up. Near $\omega \approx \omega_0$ there is a resonance. Notice that n also has an imaginary part here, and it has the correct sign:

$$\mathrm{Im}\, n > 0 \Rightarrow \mathrm{Im}\, k > 0 \Rightarrow \text{exponential damping in } z$$

$$\mathrm{Re}\, n = 1 + \frac{e^2 \rho}{2\epsilon_0 m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}$$

$$\mathrm{Im} = \frac{e^2 \rho}{2\epsilon_0 m} \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}$$



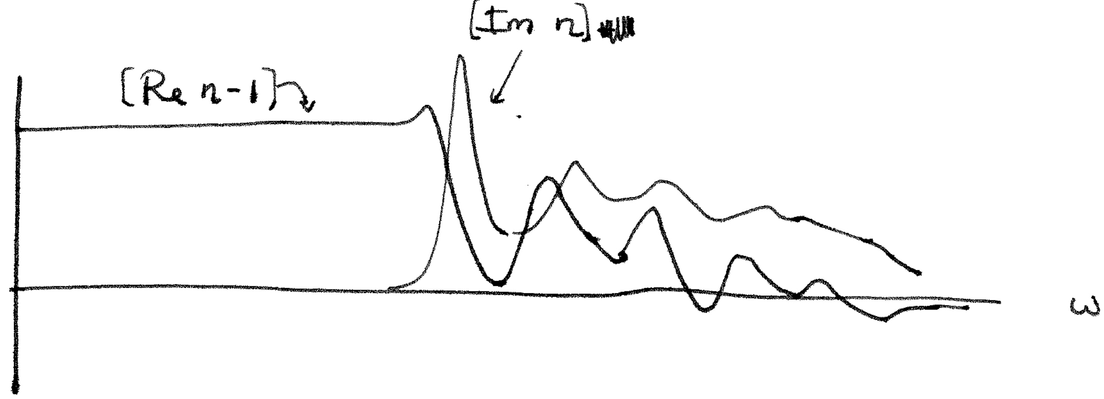
$$(\omega - \omega_0) \sim \gamma \sim 1/\tau$$

This is the typical behavior of n or k/ω in the vicinity of an atomic transition. The behavior

$\frac{dn}{d\omega} < 0$ or $\frac{d\omega}{dk} < 0$ in the region close to the resonance is called "anomalous dispersion". But, though it is "anomalous", it is the usual behavior.

Notice that the rapid change in $\mathrm{Re}\, n(\omega)$ as a function of ω is associated with the appearance of a nonzero $\mathrm{Im}\, n(\omega)$. It can be shown that this is a general phenomenon. As we saw in our general study of linear systems, a response funct like $n(\omega)$ should be an analytic funct with poles in the lower half-plane. That fact leads to relations between $\mathrm{Re}\, n(\omega)$ and $\mathrm{Im}\, n(\omega)$, called Kramers - Kronig relations.

For a real gas, there are numerous resonances, and so we have



with both $(\mathrm{Re}\, n-1)$, $\mathrm{Im}\, n \to 0$ for $\hbar\omega \gg 10\,\mathrm{eV}$.