# Waves in Less Ideal Media

Feb. 21

In the previous lecture, we studied electromagnetic waves in ideal polarizable media. I would now like to consider a few effects ~~that~~ appear when the media are less ideal.

To begin, let's return to the problem of reflection of electromagnetic waves from a conductor. Last time, we assumed a perfect conductor, which responded instantaneously to an imposed $\vec{E}$ field. Now I would like to consider an imperfect conductor — a material with finite (perhaps large) conductivity. When an $\vec{E}$ field is imposed on such a medium, a current flows such that

$$\vec{J} = \sigma \vec{E} \qquad \sigma = \text{conductivity}$$

It's consistent to assume ~~that~~ charge does not accumulate:

$$\rho = 0 \Rightarrow \vec{\nabla}\cdot\vec{J} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{E} = 0$$

Maxwell's equations now take the form

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \sigma\vec{E} + \frac{\partial \vec{D}}{\partial t}$$

I will continue to set $\vec{D} = \epsilon\vec{E}$ $\quad \vec{B} = \frac{1}{\mu}\vec{H}$, $\epsilon, \mu$ constant.

Let's look for wave solutions of these equations

$$\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{\partial}{\partial t} \frac{1}{\mu}(\vec{\nabla}\times\vec{B}) - \sigma \frac{\partial E}{\partial t}$$

$$= \frac{1}{\mu} \nabla \times (-\vec{\nabla}\times\vec{E}) - \sigma \frac{\partial E}{\partial t}$$

using $\vec{\nabla}\cdot\vec{E} = 0$ as before

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\epsilon\mu} \nabla^2 \vec{E} + \frac{\sigma}{\epsilon} \frac{\partial \vec{E}}{\partial t} = 0$$

with $\vec{\nabla}\cdot\vec{E} = 0$ $\quad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$ as before

try a solution

$$\vec{E} = \mathrm{Re}\; E_0 \vec{\epsilon}\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

Plugging this into the wave equation:

$$0 = \mathrm{Re}\; E_0 \vec{\epsilon}\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \left\{ -\omega^2 + \frac{1}{\epsilon\mu} k^2 - i\omega \frac{\sigma}{\epsilon} \right\} = 0$$

We recognize $\bar{c}^2 = \frac{1}{\epsilon\mu}$ so

$$\bar{c}^2 k^2 = \omega^2 + i\frac{\sigma}{\epsilon}\omega$$

$$k = \frac{1}{\bar{c}}\, \omega \left(1 + i\, \frac{\sigma/\varepsilon}{\omega}\right)^{1/2}$$

The factor $\varepsilon/\sigma$ in this equation has a definite physical significance.

Write

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j} = 0$$

and put in $\rho = \epsilon \vec{\nabla}\cdot\vec{E}$ , $\vec{j} = \sigma\vec{E}$ . Then

$$\frac{\partial}{\partial t}\, \epsilon(\vec{\nabla}\cdot\vec{E}) = -\sigma(\vec{\nabla}\cdot\vec{E})$$

so $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon} \simeq \frac{\rho}{\epsilon}\Big|_{t=0} e^{-t/\tau}$ where $\tau = \varepsilon/\sigma$

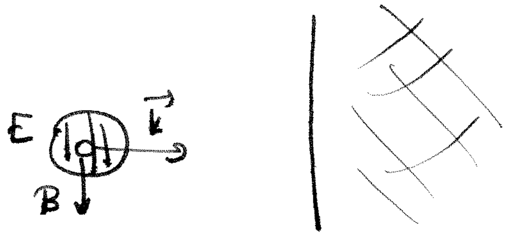
$\tau = \epsilon/\sigma$ is the time required for charge or E field to dissipate due to the effect of the current. It is natural that $\tau \to 0$ as $\sigma \to \infty$. The relation between k and ω on the previous page can be written:

$$k = \frac{\omega}{c}\left(1 + \frac{i}{\omega\tau}\right)^{1/2}$$

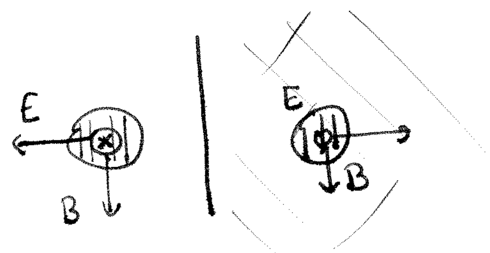
Now we can study this in two limits. First, $\omega\tau \gg 1$. This is the limit of low conductivity $\sigma \to 0$, or the limit in which the $\vec{E}$ field is oscillating too fast for the current to compensate it. In this limit

$$k = \frac{\omega}{c} + \frac{1}{2}\frac{i}{c\tau} + \cdots$$

Let's apply this formula in a practical situation. Consider an electromagnetic wave incident on a medium with small $\sigma$




Some of the wave is reflected, some is transmitted




The transmitted wave has the same frequency $\omega$ as the incident wave and a wavenumber $k$ determined by the properties of the medium. So, this wave is:

$$\vec{E}(t,\vec{x}) = \mathrm{Re}\; E_0 \vec{\epsilon}\, T \;\; e^{ikz - i\omega t}$$

where $$k = \frac{\omega}{\bar{c}} + \frac{i}{2}\frac{1}{\bar{c}\tau} + \dots$$

$$= \mathrm{Re}\left\{ E_0 \vec{\epsilon}\, T \; e^{i\frac{\omega}{\bar{c}}(z - \bar{c}t)} \right\} \cdot e^{-\frac{1}{2}\frac{z}{\bar{c}\tau}}$$

The wave falls off exponentially as it goes into the medium.

Energy density $\mathcal{E}(z) \sim e^{-z/\bar{c}\tau}$

Notice that, physically, we must have $\underline{\mathrm{Im}\, k > 0}$.

The loss of energy is due to dissipation from currents moving in the resistive media.

The $\vec{B}$ field in this wave is given by

$$\frac{\partial}{\partial t}\vec{B} = -\vec{\nabla}\times\vec{E}$$

so that, as before:

$$\vec{B} = \mathrm{Re}\ \frac{\vec{k}}{\omega}\times E_0\vec{\epsilon}T\ e^{ikz-i\omega t}$$

But, now $\vec{k}$ is complex, and so

$$\vec{B} = \mathrm{Re}\ \frac{1}{c}\left(1+\frac{i}{2\omega\tau}+\dots\right)\hat{z}\times\vec{\epsilon}\ E_0\vec{\epsilon}T\ e^{ikz-i\omega t}$$

Taking $E_0T$ real for simplicity

$$\vec{E} = \vec{\epsilon}\ E_0T\ \cos\left(\frac{\omega}{c}(z-\bar{c}t)\right)e^{-z/\bar{c}\tau}$$

$$\vec{B} = \hat{z}\times\vec{\epsilon}\ \frac{E_0T}{\bar{c}}\left[\cos\left(\frac{\omega}{c}(z-\bar{c}t)\right)-\frac{1}{2\omega\tau}\sin\frac{\omega}{c}(z-\bar{c}t)\right]e^{-z/\bar{c}\tau}$$

so $\vec{E}$ and $\vec{B}$ are slightly out of phase.

Now consider the opposite limit $\omega\tau \ll 1$. This is the limit of <u>low</u> frequency or high cond<u>uct</u>ivity ($\sigma\to\infty$), the limit in which the currents can keep up with the oscillating waves. Then

$$k = \frac{1}{c}\left(i\frac{\omega}{\tau}\right)^{\frac{1}{2}}$$

since $(i)^{\frac{1}{2}} = \frac{1+i}{\sqrt{2}}$ (We take the square root that gives $\mathrm{Im}\, k > 0$)

$$k = \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{\omega}{\bar{c}^2\tau}\right)^{\frac{1}{2}}$$

The solution for $\vec{E}$ and $\vec{B}$ is

$$\vec{E} = \mathrm{Re}\; E_0 \vec{\epsilon}\; e^{-i\omega t}\, e^{i\left(\frac{\omega}{2\bar{c}^2\tau}\right)^{\frac{1}{2}} z}\; e^{-\left(\frac{\omega}{2\bar{c}^2\tau}\right)^{\frac{1}{2}} z}$$

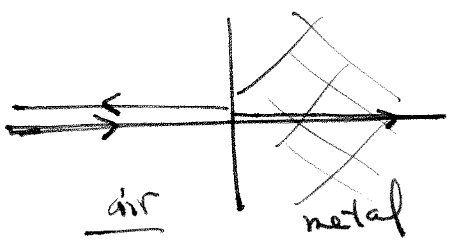
$$\vec{B} = \mathrm{Re}\; \frac{E_0}{\bar{c}}\, \hat{z}\times\vec{\epsilon}\, \left(\frac{1+i}{(2\omega\tau)^{\frac{1}{2}}}\right) e^{-i\omega t}\, e^{i\left(\frac{\omega}{2\bar{c}^2\tau}\right)^{\frac{1}{2}} z}\; e^{-\left(\frac{\omega}{2\bar{c}^2\tau}\right)^{\frac{1}{2}} z}$$

Now the $\vec{E}$ and $\vec{B}$ fields are 45° out of phase. The waves fall off exponentially in the distance

$$\delta = \left[\frac{2\bar{c}^2\tau}{\omega}\right]^{\frac{1}{2}} = \left(\frac{2}{\mu\sigma\omega}\right)^{\frac{1}{2}}$$

called the "skin depth". As $\sigma \to \infty$, $\delta \to 0$. The electric and magnetic fields are present only in a small surface layer of the conductor.

Let's solve the problem of reflection at normal incidence in this limit. ~~The~~ solution to Maxwell's equation is



$$\vec{E} = \begin{cases} \mathrm{Re}\ E_0 \vec{\epsilon}\left(e^{ik_0 z - i\omega t} + R\, e^{-ik_0 z - i\omega t}\right) & z<0 \\ \mathrm{Re}\ E_0 \vec{\epsilon}\, T\, e^{ikz - i\omega t} & z>0 \end{cases}$$

$$\vec{B} = \begin{cases} \mathrm{Re}\ \frac{\hat{z}\times\vec{\epsilon}}{c} E_0 \left(e^{ik_0 z - i\omega t} - R\, e^{-ik_0 z - i\omega t}\right) & z<0 \\ \mathrm{Re}\ E_0\, \hat{z}\times\vec{\epsilon}\, T \frac{k}{\omega}\, e^{ikz - i\omega t} & z>0 \end{cases}$$

where $\quad k_0 = \frac{\omega}{c} \qquad k = \frac{1+i}{\sqrt{2}}\left(\frac{\omega}{\bar{c}^2 \tau}\right)^{1/2}$

$$E_{\parallel}|_1 = E_{\parallel}|_2 \Rightarrow \quad 1 + R = T$$

$$H_{\parallel}|_1 = H_{\parallel}|_2 \Rightarrow \quad \frac{1}{\mu_0 c}(1-R) = \frac{1}{\mu}\frac{k}{\omega} T$$

the solution is

$$R = \frac{1 - \frac{\mu_0}{\mu}\frac{ck}{\omega}}{1 + \frac{\mu_0}{\mu}\frac{ck}{\omega}} \qquad T = \frac{2}{1 + \frac{\mu_0}{\mu}\frac{ck}{\omega}}$$

Now $\quad \frac{\mu_0}{\mu}\frac{ck}{\omega} = \frac{1+i}{\sqrt{2}}\frac{\mu_0}{\mu}\left(\frac{c^2}{\bar{c}^2}\right)^{1/2}\frac{1}{(\omega\tau)^{1/2}}$

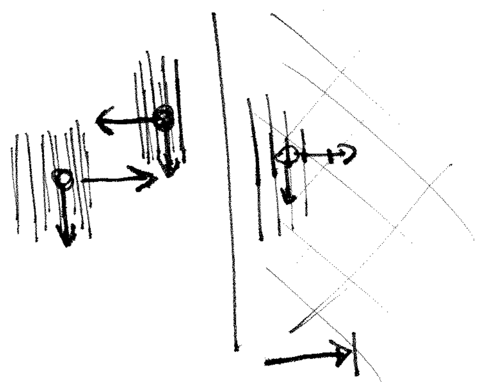
$$= \frac{1+i}{\sqrt{2}}\left[\frac{\mu_0}{\mu\epsilon_0}\frac{\sigma}{\omega}\right]^{1/2}$$

We are working in the limit $\omega\tau \to \infty$, $\omega \to 0$ or $\sigma \to \infty$.
In this limit, $(\sigma/\omega)^{1/2} \to \infty$, so

$$R = -1 + 2\,\frac{1-i}{\sqrt{2}}\left[\frac{\mu\epsilon_0}{\mu_0}\,\frac{\omega}{\sigma}\right]^{1/2} + \cdots$$

$$T = 2\,\frac{1-i}{\sqrt{2}}\left[\frac{\mu\epsilon_0}{\mu_0}\,\frac{\omega}{\sigma}\right]^{1/2} + \cdots$$

$R \to -1$, $T \to 0$, and note that this last bit of transmitted wave is $45^\circ$ out of phase with the initial wave.



$\delta$ skin depth