Feb. 14

Reflection, Transmission, Refraction - 2

In the last lecture, we studied the transmission of a wave in one dimension through a boundary. Let's now generalize this, first to a three dimensional problem and then to electromagnetism.

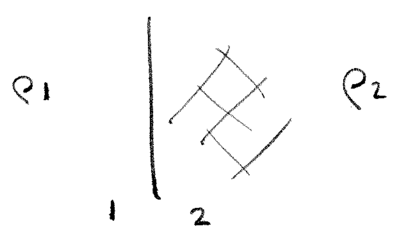
First, imagine a boundary between elastic media in three dimensions. For a system with

$$E = \int d^3x \left[\frac{1}{2} \rho(x) \left(\frac{\partial \chi}{\partial t} \right)^2 + \frac{1}{2} \kappa \left(\vec{\nabla} \chi \right)^2 \right]$$

Let

$$\rho = \begin{cases} \rho_1 & z < 0 \\ \rho_2 & z > 0 \end{cases}$$

and κ = constant and equal on the two sides.



As before,

$$c_1 = \sqrt{\frac{\kappa}{\rho_1}} \qquad c_2 = \sqrt{\frac{\kappa}{\rho_2}}$$

are the wave velocities in the two media. If we impose the boundary conditions at the junction:

$$\chi\big|_1 = \chi\big|_2 \qquad \hat{n}\cdot\vec{\nabla}\chi\big|_1 = \hat{n}\cdot\vec{\nabla}\chi\big|_2$$

then energy is guaranteed to be conserved. Let's look for wave solutions in this geometry. In particular, we will look for a solution corresponding to a wave packet coming in from the left and then being reflected or transmitted through the boundary.

A right-moving wave solution for $z<0$ has the form

$$\chi(t,\vec{x}) = \mathrm{Re}\left\{ \chi_0\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \right\}$$

with $\omega = c_1 |\vec{k}|$ and $k_z > 0$. Write

$$\vec{k} = (k_x, k_y, k_z) = (\vec{k}_\perp, k_z)$$

The case $\vec{k}_\perp = 0$ gives the formulae of the previous lecture, so keep $\vec{k}_\perp \neq 0$. Let

$$\vec{k}_R = (k_\perp, -k_z)$$

$$\vec{k}_T = (k_\perp, k_{zT})$$ for some k_{zT} that we must find

Then I claim that we can find a solution which respects the boundy conditions which is of the form:

$$\chi_{\vec{k}}(t,\vec{x}) = \begin{cases} \mathrm{Re}\left\{ \chi_0 \left(e^{i\vec{k}\cdot\vec{x} - i\omega t} + R\, e^{i\vec{k}_R\cdot\vec{x} - i\omega t} \right) \right\} & z<0 \\ \mathrm{Re}\left\{ \chi_0 T\, e^{i\vec{k}_T\cdot\vec{x} - i\omega t} \right\} & z>0 \end{cases}$$

Check: at $z=0$

$$\chi_k(t,x,y,z=0) = \chi_0 \left(e^{i\vec{k}_\perp\cdot\vec{x}_\perp} e^{-i\omega t} + R\, e^{i\vec{k}_\perp\cdot\vec{x}_\perp - i\omega t} \right) \leftarrow ①$$

$$\vec{x}_\perp = (x,y) \qquad = \chi_0 T\, e^{i\vec{k}_\perp\cdot\vec{x}_\perp - i\omega t} \leftarrow ②$$

$$\frac{\partial \chi_k}{\partial z}(t,x,y,z=0) = \chi_0\,(ik_z)\left(e^{i\vec{k}_\perp\cdot\vec{x}_\perp} e^{-i\omega t} - R\, e^{ik_\perp x_\perp - i\omega t} \right) \leftarrow ①$$

$$= \chi_0 T\, ik_{zT}\, e^{i\vec{k}_\perp \vec{x}_\perp - i\omega t} \leftarrow ②$$

so we can find a solution simply by solving the equations

$$1 + R = T$$

$$k_z\,(1-R) = k_{zT}\, T$$

for R and T.

what is k_{zT}? In medium 2:

$$|\vec{k}_2|^2 = |k_\perp^2 + k_{zT}^2| = \frac{1}{c_2^2}\omega^2 = \frac{c_1^2}{c_2^2}(k_\perp^2 + k_z^2)$$

or

$$k_{zT} = \left[\left(\frac{c_1^2}{c_2^2} - 1\right) k_\perp^2 + \frac{c_1^2}{c_2^2} k_z^2\right]^{1/2}$$

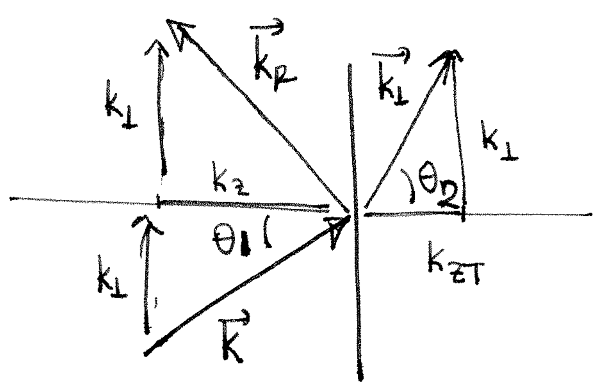
This formula is easier to understand if we recast it in the follow way:

$$\frac{1}{c_1}\left|\frac{\vec{k}_\perp}{\vec{k}}\right| = \frac{1}{c_2}\left|\frac{\vec{k}_\perp}{\vec{k}_{\mathrm{II}}}\right|$$

let c_0 be the wave speed in some reference medium, and define

$$n_1 = \frac{c_0}{c_1} \qquad n_2 = \frac{c_0}{c_2}$$

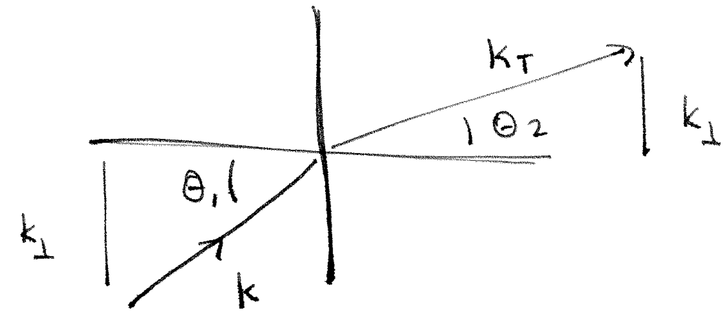
Define the angles θ_1, θ_2 by the construction:



then

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

This relation is called Snell's law. If $\rho_2 > \rho_1$, then $c_2 < c_1$, $n_2 > n_1$. In this case, $\vec{k}_T$ is at a smaller angle to the normal than $\vec{k}$:

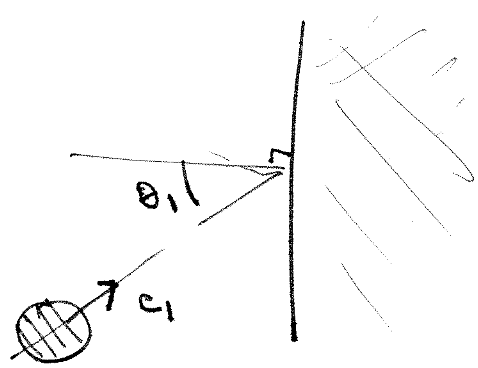


since $$|\vec{k}_T| = \frac{c_1}{c_2}|\vec{k}| > |\vec{k}|$$

With this information, we can complete the solution:

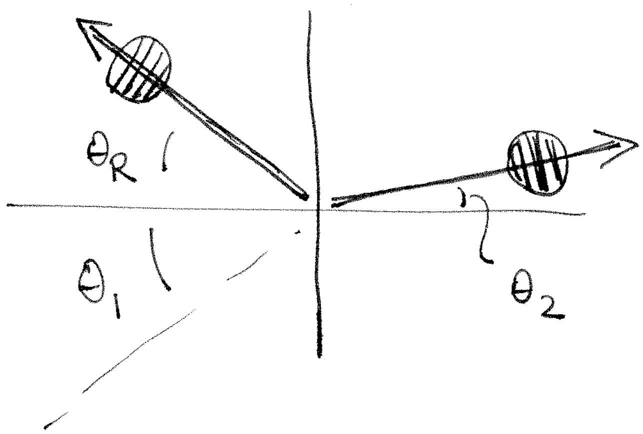
$$R = \frac{k_z - k_{zT}}{k_z + k_{zT}} \qquad T = \frac{2k_z}{k_z + k_{zT}}$$

The solution has the following physical interpretation: If we form a wavepacket far to the left, then at early times the solution consists of this wavepacket, moving toward the join:



θ_1 is also called θ_i, the angle of incidence.

At late times, the solution consists of two wave packets, one on the left moving with speed c_1 in the direction $\hat{k}_R$, and one on the right moving with speed c_2 in the direction $\hat{k}_T$:



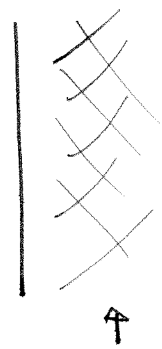
with $\quad \theta_1 = \theta_R \quad$ angle of incidence = angle of reflection

$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad$ Snell's law of refraction

θ_2 is also called θ_r, the angle of refraction.

We have now seen that reflection and refraction — familiar properties of light at interfaces — are found quite generally in systems with waves. Now, can we find these phenomena in the waves associated with Maxwell's equations?

I'll begin with the two types of ideal media that we studied last term — a perfect conductor and a linear polarizable medium. Let's first study electromagnetic waves outside a perfect conductor:



(vacuum)

$\vec{E} = \vec{B} = 0$ inside

Since there can be surface charges and currents, we cannot say anything definite about $E_\perp$ or $B_\parallel$ at the interface. However, the other two Maxwell equations imply:

$$\vec{E}_\parallel = 0 \qquad B_\perp = 0 \quad \text{at} \quad z = 0.$$

We can look for a wave solution in the region $z<0$ that satisfies these conditions.

The general form of the wave we are looking for is

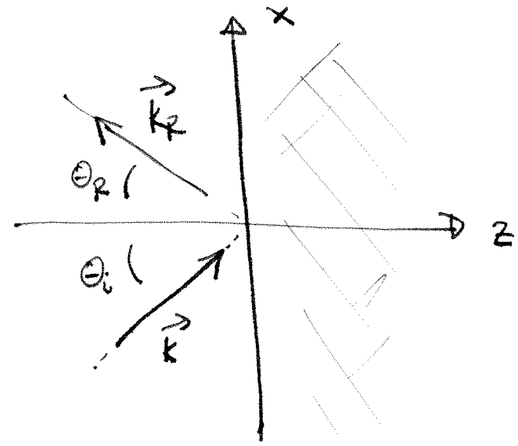
$$\vec{E}(t,\vec{x}) = \mathrm{Re}\left\{ \vec{\epsilon}\, E_0\, e^{i\vec{k}\cdot\vec{x} - i\omega t} + E_0 \vec{R}\, e^{i\vec{k}_R\cdot\vec{x} - i\omega t} \right\}$$

$$\vec{B}(t,\vec{x}) = \mathrm{Re}\left\{ \hat{k}\times\vec{\epsilon}\, \frac{E_0}{c}\, e^{i\vec{k}\cdot\vec{x} - i\omega t} + \frac{E_0}{c}\, \hat{k}\times\vec{R}\, e^{i\vec{k}_R\cdot\vec{x} - i\omega t} \right\}$$

There are two cases to consider. Let the incoming wave have $\vec{k}$ in the $(\hat{x}, \hat{z})$ plane

$$\vec{k} = (k_x, 0, k_z)$$

$$\vec{k}_R = (k_x, 0, -k_z)$$



Then we can have

(II) polarization parallel to the plane of incidence:
$\vec{\varepsilon}$ in the xz plane $\qquad \vec{k}\times\vec{\varepsilon} \parallel \hat{y}$

($\perp$) polarization perpendicular to the plane of incidence:
$\vec{\varepsilon} = \hat{y} \qquad \vec{k}\times\vec{\varepsilon}$ in the x, z plane.

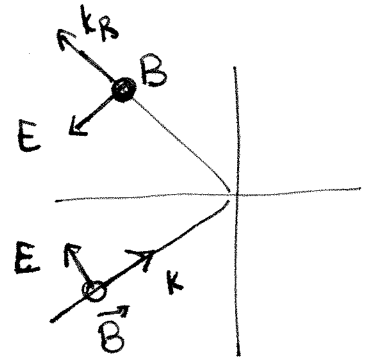
[the general case is a linear superposition of these]

Analyze each case in turn:

(II): $$\vec{\varepsilon}_k = \frac{(k_z, 0 \; -k_x)}{\sqrt{k_x^2 + k_z^2}} \qquad \vec{k}\times\vec{\varepsilon} = \hat{y}$$

for the reflected wave, take

$$\vec{\varepsilon}_R = \frac{(-k_z \; 0 \; -k_x)}{\sqrt{k_x^2 + k_z^2}} \qquad \vec{k}_R\times\vec{\varepsilon}_R = \hat{y}$$



then we can see that

$$\vec{E}(t,\vec{x}) = \mathrm{Re}\ E_0 \left(\vec{\varepsilon}_k e^{i\vec{k}\vec{x} - i\omega t} + \vec{\varepsilon}_R e^{i\vec{k}_R \cdot \vec{x} - i\omega t} \right)$$

$$\vec{B}(t,\vec{x}) = \mathrm{Re}\ \tfrac{1}{c} E_0 \left(\hat{y}\, e^{i\vec{k}\cdot\vec{x} - i\omega t} + \hat{y}\, e^{i\vec{k}_R \cdot \vec{x} - i\omega t} \right)$$

satisfies our conditions:

$$\vec{E}_x = 0 \qquad \vec{B}_z = 0 \quad \text{at} \quad z = 0$$

$\perp$ We can reuse the vectors $\vec{\varepsilon}_k, \vec{\varepsilon}_R$ as follows.

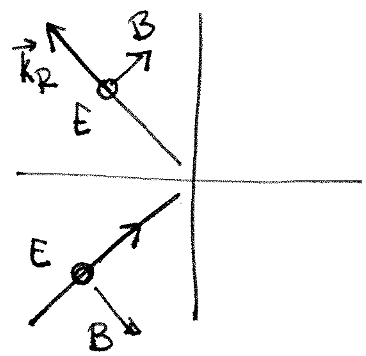
$$\vec{E}(t,\vec{x}) = \mathrm{Re}\ E_0 \left[\hat{y}\, e^{i\vec{k}\cdot\vec{x} - i\omega t} - \hat{y}\, e^{i\vec{k}_R \cdot \vec{x} - i\omega t} \right]$$

$$\vec{B}(t,\vec{x}) = \mathrm{Re}\ \frac{E_0}{c} \left[-\vec{\varepsilon}_k e^{i\vec{k}\cdot\vec{x} - i\omega t} + \vec{\varepsilon}_R e^{i\vec{k}_R \cdot \vec{x} - i\omega t} \right]$$

this solution satisfies the conditions

$$\vec{E}_y = 0 \qquad \vec{B}_z = 0$$

at $z = 0$



In both cases, the solution has the following physical interpretation: A wave packet traveling with velocity $c\hat{k}$ reflects from the wall at $z=0$ and turns into a wavepacket with velocity $c\hat{k}_R$. The amplitude of the $\vec{E}$ and $\vec{B}$ fields in the final wavepacket is equal to the amplitude in the initial wavepacket — so energy is conserved — and the phase of the ∥ components of the $\vec{E}$ field is flipped by π.

I will come back later to the case of an imperfect conductor. Let's first treat the case of an ideal dielectric.

To study the behavior of electromagnetic waves at the surface of a dielectric, we should first study the wave solutions well inside an extended polarizable medium. I will assume that the medium is linear and isotropic, so

$$\vec{D} = \epsilon \vec{E} \qquad \vec{B} = \mu \vec{H}$$

Then Maxwell's equations read (for $\rho_f = \vec{J}_f = 0$)

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t}$$

then

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\epsilon}\vec{\nabla}\times\frac{\partial H}{\partial t} = \frac{1}{\epsilon\mu}\vec{\nabla}\times(-\vec{\nabla}\times\vec{E})$$

$$= \frac{1}{\epsilon\mu}\left[-\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) + \nabla^2\vec{E}\right]$$

Using $\vec{\nabla}\cdot\vec{E} = 0$, we see that $\vec{E}$ obeys the wave equation

$$\left(\frac{1}{\bar{c}^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{E} = 0$$

where now
$$\bar{c}^2 = \frac{1}{\epsilon\mu}$$

Define

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$
the <u>index of refraction</u> of the material

then the wave speed in this material is

$$\bar{c} = \frac{c}{n}$$

So n here is just that on p. 4, using $c_0 = c$ as a reference.
For the typical case $\varepsilon/\varepsilon_0 > 1$, $\mu \simeq \mu_0$ we see: $\underline{\bar{c} < c}$.

In a polarizable medium (as on a heavy string) electromagnetic waves travel more <u>slowly</u>.

Let's work out the form of the waves in more detail

Starting from

$$\vec{E}(x,t) = \text{Re}\ E_0 \vec{\epsilon}\, e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \omega = \bar{c}|\vec{k}|$$

and using

$$\frac{\partial B}{\partial t} = -\vec{\nabla}\times\vec{E}$$

we have

$$-i\omega \vec{B} = -i\vec{k} \times \vec{E}$$

or

$$\vec{B} = \frac{1}{c} \hat{k} \times \vec{E}$$

as a check: $\frac{\partial D}{\partial t} = \vec{\nabla} \times \vec{H} \Rightarrow \frac{\partial E}{\partial t} = \frac{1}{\epsilon\mu} \vec{\nabla} \times \vec{B}$

So

$$-i\omega \vec{E} = \bar{c}^2 \, i\vec{k} \times \vec{B}$$

$$\vec{E} = -\bar{c} \, \hat{k} \times \vec{B} = -\hat{k} \times (\hat{k} \times \vec{E})$$

which is consistent if $\hat{k} \cdot \vec{E} = 0$

The general wave solution in a polarizable medium is then

$$\vec{E}(t,\vec{x}) = \mathrm{Re} \; E_0 \vec{\epsilon} \; e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\vec{B}(t,\vec{x}) = \mathrm{Re} \; E_0 \frac{1}{\bar{c}} \hat{k} \times \vec{\epsilon} \; e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

with $\omega = \bar{c} \, |\vec{k}|$.

The energy of an electromagnetic wave in a polarizable medium is, from last term

$$\mathcal{E} = \frac{1}{2} \vec{E} \cdot \vec{D} + \frac{1}{2} \vec{B} \cdot \vec{H}$$

$$= \frac{1}{2}\epsilon E^2 + \frac{1}{2\mu} B^2$$

The energy conservation law is

$$\frac{\partial}{\partial t}\mathcal{E} = \epsilon \frac{\partial \vec{E}}{\partial t}\cdot\vec{E} + \frac{1}{\mu}\frac{\partial \vec{B}}{\partial t}\cdot\vec{B}$$

$$= (\vec{\nabla}\times\vec{H})\cdot\vec{E} - (\vec{\nabla}\times\vec{E})\cdot\vec{H}$$

$$= \epsilon^{ijk}(\nabla^j H^k)E^i - \epsilon^{ijk}(\nabla^j E^k)H^i$$

$$= -\epsilon^{ijk}\nabla^i(E^j H^k)$$

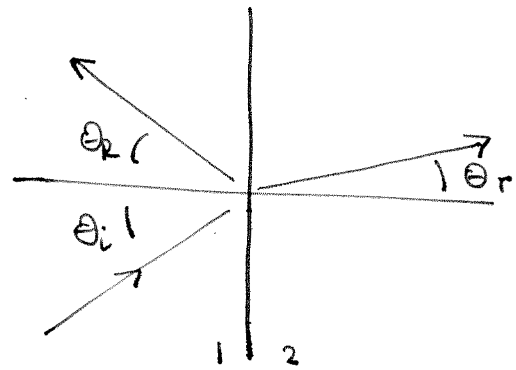
so

$$\frac{\partial}{\partial t}\mathcal{E} + \vec{\nabla}\cdot\vec{j}_{\mathcal{E}} = 0$$

where now $\quad \vec{j}_{\mathcal{E}} = \vec{E}\times\vec{H}$

In vacuum, this does reduce to $\quad \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

Now we have all of the materials we need to analyze the reflection of electromagnetic waves at an interface. Many of the general features are already clear. From our general analysis with the wave equation, we expect that a wavepacket incident on the wall at angle θ_i will turn into two packets, one reflected at an angle θ_R, the other refracting at angle θ_r



To be able to impose a boundary condition which matches the oscillations in $x_\perp$ and t on the boundary, we require

$$\theta_i = \theta_R$$

$$n_1 \sin\theta_i = n_2 \sin\theta_r$$

In particular, Maxwell's equations imply that electromagnetic waves passing from one medium to another predict that the waves obey the law of reflection and Snell's law of refraction. In addition, they allow us to compute the index of refraction $\underline{n}$ from the electrostatic and magnetostatic properties.

The only remaining question is, how much radiation is reflected, and how much is transmitted? To analyze this question, we need to take account of the boundary conditions derived from Maxwell's equations:

$$E_{\parallel}|_1 = E_{\parallel}|_2 \qquad D_\perp|_1 = D_\perp|_2 \qquad B_\perp|_1 = B_\perp|_2 \qquad H_\parallel|_1 = H_\parallel|_2$$

The strategy I will use is to set up proposed solutions for each of the two states of initial polarization described on p. 8, and then to solve for the unknown coefficients. For reference in this calculation, let me record the vectors from p. 8:

$$\hat{k} = (\sin\theta_i,\ 0,\ \cos\theta_i) \qquad \hat{k}_R = (\sin\theta_i,\ 0,\ -\cos\theta_i)$$

$$\vec{\epsilon}_k = (\cos\theta_i,\ 0,\ -\sin\theta_i) \qquad \vec{\epsilon}_R = (-\cos\theta_i,\ 0,\ -\sin\theta_i)$$

$$\hat{y} = (0, 1, 0)$$

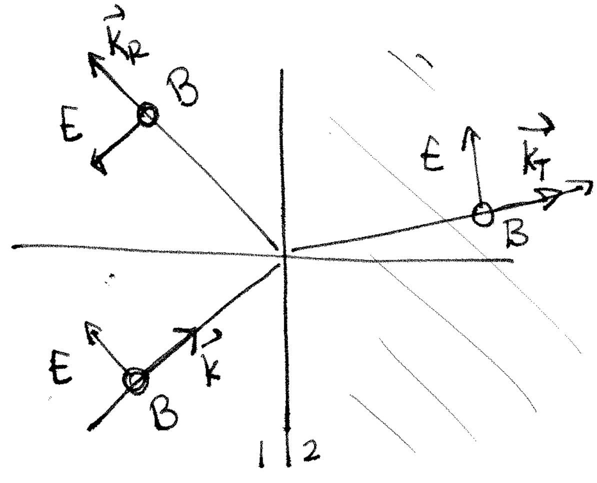
and

$$\hat{k}_T = (\sin\theta_r,\ 0,\ \cos\theta_r)$$

$$\vec{\epsilon}_T = (\cos\theta_r,\ 0,\ -\sin\theta_r)$$

Now consider the two cases on p. 8 in turn:

$\parallel$



$$\vec{E}(t,\vec{x}) = \begin{cases} \mathrm{Re}\; E_0\left(\vec{\epsilon}_k\, e^{i\vec{k}\cdot\vec{x}-i\omega t} + \vec{\varepsilon}_R \cdot R \cdot e^{i\vec{k}_R\cdot\vec{x}-i\omega t}\right) & z<0 \\ \mathrm{Re}\; E_0 T\, \vec{\varepsilon}_T\, e^{i\vec{k}_T\cdot\vec{x}-i\omega t} & z>0 \end{cases}$$

$$\vec{B}(t,\vec{x}) = \begin{cases} \mathrm{Re}\; E_0 \frac{1}{c_1}\left(\hat{y}\, e^{i\vec{k}\cdot\vec{x}-i\omega t} + \hat{y}\, R\, e^{i\vec{k}_R\cdot\vec{x}-i\omega t}\right) & z<0 \\ \mathrm{Re}\; E_0 \frac{1}{c_2} T\, \hat{y}\, e^{i\vec{k}\cdot\vec{x}-i\omega t} & \end{cases}$$

where $\quad c_1 = \dfrac{c}{n_1} \qquad c_2 = \dfrac{c}{n_2}$

The boundary conditions are:

$$E_x|_1 = E_x|_2 \Rightarrow \cos\theta_i\,(1-R) = T\cos\theta_r$$

$$\epsilon_1 E_z|_1 = \epsilon_2 E_z|_2 \Rightarrow -\epsilon_1 \sin\theta_i\,(1+R) = -\epsilon_2\, T \sin\theta_r$$

$$B_z|_1 = B_z|_2 \Rightarrow 0 = 0 \quad \checkmark$$

$$\frac{1}{\mu_1} B_y|_1 = \frac{1}{\mu_2} B_y|_2 \Rightarrow \frac{1}{\mu_1 c_1}(1+R) = \frac{1}{\mu_2 c_2} T$$

since $\quad \dfrac{1}{\mu_1 c_1} = \dfrac{\epsilon_1}{\epsilon_1 \mu_1 c_1} = \epsilon_1 c_1 = \epsilon_1 \dfrac{c}{n_1}$

and similarly for $\mu_2 c_2 \varepsilon_2$, $\quad \dfrac{1}{\mu_1 c_1} : \dfrac{1}{\mu_2 c_2} = \dfrac{\epsilon_1}{n_1} : \dfrac{\epsilon_2}{n_2} = \epsilon_1 \sin\theta_i : \epsilon_2 \sin\theta_r$

so the last equation is redundant with the second.

Now we can solve the first two equations for R and T:

$$(1-R) = T\left(\frac{\cos\theta_r}{\cos\theta_i}\right)$$

$$(1+R) = T\left(\frac{\epsilon_2 \sin\theta_r}{\epsilon_1 \sin\theta_i}\right) = T\left(\frac{\epsilon_2}{n_2}\frac{n_1}{\epsilon_1}\right) = T\left(\frac{\mu_1 n_2}{\mu_2 n_1}\right)$$

using $\dfrac{n_1}{n_2} = \left(\dfrac{\epsilon_2\mu_2}{\epsilon_1\mu_1}\right)^{1/2}$ in the last step.

then

$$R = \left(\frac{\alpha-\beta}{\alpha+\beta}\right) \qquad T = \frac{2}{\alpha+\beta}$$

Where

$$\alpha = \frac{\cos\theta_r}{\cos\theta_i} \qquad \beta = \frac{\mu_1 n_2}{\mu_2 n_1}$$

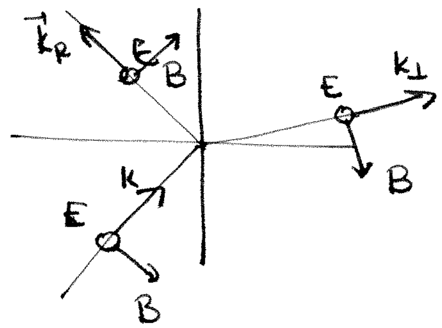
(For the case of non-magnetic materials, $\mu_1/\mu_2 = 1$ and

$$\beta = \left(\frac{\sin\theta_i}{\sin\theta_r}\right) = \frac{n_2}{n_1}.)$$

then

$$R = \left(\frac{\cos\theta_r - \beta\cos\theta_i}{\cos\theta_r + \beta\cos\theta_i}\right) \qquad T = \left(\frac{2\cos\theta_i}{\cos\theta_r + \beta\cos\theta_i}\right)$$

$\perp$



$$\vec{E}(t,x) = \begin{cases} \mathrm{Re}\; E_0\left(\hat{y}\, e^{i\vec{k}\cdot\vec{x}-i\omega t} + \hat{y}\, R\, e^{i\vec{k}_R\cdot\vec{x}-i\omega t}\right) & z<0 \\ \mathrm{Re}\; E_0\, T\, \hat{y}\, e^{i\vec{k}_T\cdot\vec{x}-i\omega t} & z>0 \end{cases}$$

$$\vec{B}(t,x) = \begin{cases} \mathrm{Re}\; \frac{E_0}{c_1}\left(-\vec{\varepsilon}_k\, e^{i\vec{k}\cdot\vec{x}-i\omega t} - \vec{\varepsilon}_R\, R\, e^{i\vec{k}_R\cdot\vec{x}-i\omega t}\right) & z<0 \\ \mathrm{Re}\; \frac{E_0}{c_2}\, T\left(-\vec{\varepsilon}_T\, e^{i\vec{k}_T\cdot\vec{x}-i\omega t}\right) & z>0 \end{cases}$$

The boundary conditions are:

$E_y|_1 = E_y|_2 \Rightarrow \quad (1+R) = T$

$\epsilon E_z|_1 = \epsilon E_z|_2 \Rightarrow \quad 0 = 0 \;\checkmark$

$B_z|_1 = B_z|_2 \Rightarrow \quad \frac{1}{c_1}\sin\theta_i\,(1+R) = \frac{1}{c_2}\sin\theta_r\, T$

$\frac{1}{\mu}B_x|_1 = \frac{1}{\mu}B_x|_2 \Rightarrow \quad \frac{1}{\mu_1 c_1}\cos\theta_i\,(1-R) = \frac{1}{\mu_2 c_2}\, T\cos\theta_r$

since $\frac{1}{c_1} = \frac{n_1}{c}$, the third equation is redundant with the first.

the remaining two equations are:

$$(1 + R) = T$$

$$(1 - R) = T \frac{\mu_1 c_1}{\mu_2 c_2} \frac{\cos\theta_r}{\cos\theta_i} = T \beta \frac{\cos\theta_r}{\cos\theta_i}$$

$$R = \frac{\cos\theta_i - \beta\cos\theta_r}{\cos\theta_i + \beta\cos\theta_r} \qquad T = \left(\frac{2\cos\theta_i}{\cos\theta_i + \beta\cos\theta_r} \right)$$

the relations:

$$\beta = \frac{\mu_1 c_1}{\mu_2 c_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

$$\parallel \qquad R = \frac{\cos\theta_r - \beta\cos\theta_i}{\cos\theta_r + \beta\cos\theta_i} \qquad T = \frac{2\cos\theta_i}{\cos\theta_r + \beta\cos\theta_i}$$

$$\perp \qquad R = \frac{\cos\theta_i - \beta\cos\theta_r}{\cos\theta_i + \beta\cos\theta_r} \qquad T = \frac{2\cos\theta_i}{\cos\theta_i + \beta\cos\theta_r}$$

are called Fresnel's equations. Notice that, in both cases, R and T satisfy

$$1 = |R|^2 + \beta \frac{\cos\theta_r}{\cos\theta_i} |T|^2$$

I would now like to demonstrate that this is the constraint of energy conservation. This is easiest to see by computing the flux of energy through a unit area placed perpendicular to $\hat{z}$:

$$\left\langle \frac{\Phi_\varepsilon}{\text{Area}} \right\rangle = \langle \hat{n} \cdot \vec{J}_\varepsilon \rangle = \langle \hat{n} \cdot (\vec{E} \times \vec{H}) \rangle$$

$$= \frac{1}{\mu} \hat{n} \cdot \langle \vec{E} \times \vec{B} \rangle$$

then,

$$= \frac{1}{\mu c} \cos\theta \, |E_0|^2 \cdot \frac{1}{2}$$

for the incident wave:

$$\left\langle \frac{\Phi_\varepsilon}{\text{Area}} \right\rangle = \frac{1}{2} \cos\theta_i \frac{1}{\mu_1 c_1} |E_0|^2$$

for the reflected wave:

$$\left\langle \frac{\Phi_\varepsilon}{\text{Area}} \right\rangle = \frac{1}{2} \cos\theta_i \frac{1}{\mu_1 c_1} |R|^2 |E_0|^2$$

for the refracted wave:

$$\left\langle \frac{\Phi_\varepsilon}{\text{Area}} \right\rangle = \frac{1}{2} \cos\theta_r \frac{1}{\mu_2 c_2} |T|^2 |E_0|^2$$

so if the same amount of energy goes out, away from the interface, as comes in

$$\cos\theta_i \cdot 1 = \cos\theta_i |R|^2 + \frac{\mu_1 c_1}{\mu_2 c_2} \cos\theta_r |T|^2$$

which is exactly the relation on p. 19. The quantity $|R|^2$ is the fraction of the incident energy which is reflected rather than transmitted.

Look again at Fresnel's equations, specializing to the case where the two media are dielectrics, with $n_1 < n_2$ (e.g. 1 = air, 2 = glass). Then

$$\beta = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} > 1$$

at the same time

$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1} > 1$$

$$\frac{\cos\theta_i}{\cos\theta_r} < 1$$

This means that the numerator of R in the $\perp$ case is negative definite:

$$\beta\cos\theta_r > \cos\theta_i$$

so $|R|^2$ never vanishes in this case. However, for the $\parallel$ case, $|R|^2$ does vanish at a special angle where

$$\cos\theta_r = \beta\cos\theta_i$$

The θ_i in this case is called Brewster's angle. θ_B

$$\cos^2\theta_r = \left(\frac{n_2}{n_1}\right)^2 \cos^2\theta_B$$

$$\left(\frac{n_1}{n_2}\right)^2 (1 - \sin^2\theta_r) = 1 - \sin^2\theta_B$$

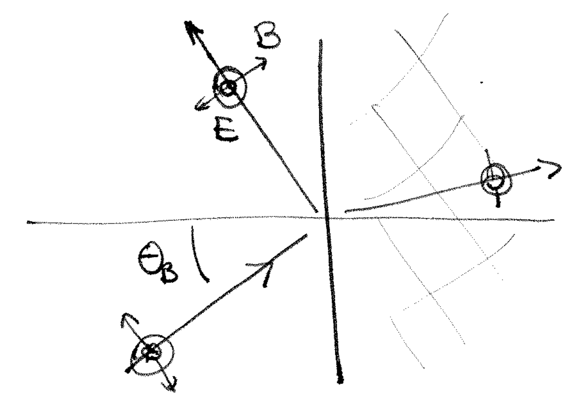
and, using Snell's law

$$\left(\frac{n_1}{n_2}\right)^2 \left(1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B\right) = 1 - \sin^2\theta_B$$

After some reductions, this becomes

$$\sin^2\theta_B = \frac{1}{1+(n_1/n_2)^2} \quad \text{or} \quad \tan\theta_B = \frac{n_2}{n_1}$$

At this special angle of incidence, if a randomly polarized beam of light impinges on a surface, the reflected beam is perfectly polarized $\perp$ to the plane of incidence:



Note that, near θ_B, $|R_\parallel|^2 \sim (\theta-\theta_B)^2$. This is why polaroid sunglasses are useful.