Feb. 12

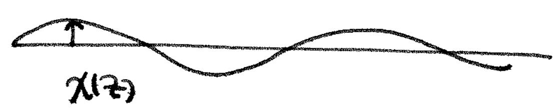
Reflection, Transmission, Refraction

In the last lecture, I introduced the scalar wave equation in one dimension. In this lecture, I would like to do some exercises with this equation, preparatory to discussing reflection and refraction of electromagnetic waves.

For concreteness, think about waves on a string. Each bit of string Δz has mass $\rho \Delta z$, where ρ is the density of the string material. The kinetic energy is then

$$\frac{1}{2}(\dot{\chi})^2 \rho \Delta z$$

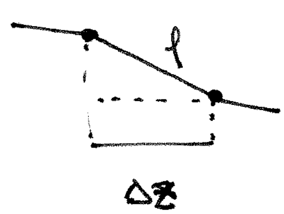
where χ is the transverse displacement, summed over bits.



so

$$T = \int dz \ \frac{1}{2} \rho (\dot{\chi})^2$$

If the string is stretched from its natural position



there is a force $-k\,\vec{\Delta l}$ along the direction of stretching.

This leads to a potential energy

$$V = \tfrac{1}{2} k\, l^2 = \tfrac{1}{2} k \left[(\Delta z)^2 + \left(\frac{\partial \chi}{\partial z}\right)^2 (\Delta z)^2 \right]$$

The transverse part of this is of the form

$$V = \int dz \ \tfrac{1}{2} \kappa \left(\frac{\partial \chi}{\partial z}\right)^2$$

where $\kappa = k\Delta z$ = force (tension) in the string at equilibrium. In all

$$E = \int dz \left[\tfrac{1}{2}\rho \left(\frac{\partial \chi}{\partial t}\right)^2 + \tfrac{1}{2} \kappa \left(\frac{\partial \chi}{\partial z}\right)^2 \right]$$

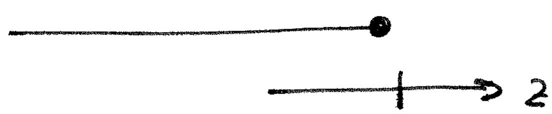
This form of the energy leads to waves which move with

$$c = \sqrt{\frac{\kappa}{\rho}}$$

and obey the wave equation

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \chi - \frac{\partial^2}{\partial z^2} \chi = 0$$

Now consider a string tied down at $z = 0$:

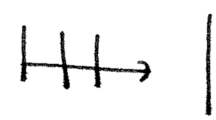


We would like to find the behavior of waves propagating in the region $z<0$ with the boundary condition

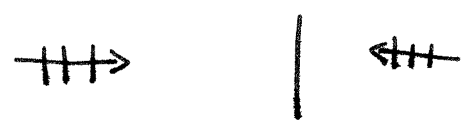
$$\chi(t, z=0) = 0$$

This is a perfectly good Dirichlet boundary condition, so we should find a unique solution for a fixed initial condition. To construct these solutions, let's find wave solutions satisfying this boundary condition. The basic wave

$$\chi(t,z) = \mathrm{Re}\left[\chi_0 \, e^{ikz - i\omega t}\right]$$



does not satisfy the correct boundary condition. But we can fix this by the method of images, adding an image wave that propagates to the left:



$$\chi(t,z) = \mathrm{Re}\left[\chi_0 \left(e^{ikz - i\omega t} - e^{-ikz - i\omega t}\right)\right]$$

These waves now satisfy the boundary condition for any $k>0$ and $\omega = ck$.

The physical interpretation of this solution may not be so obvious. To see it, let's construct a wavepacket centered on $z = -L_1$, $L_1 \gg a$, and let it evolve.

The initial condition is:

$$\chi(t=0, z) = \chi_0 \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(z+L_1)}\, e^{-\frac{1}{2}a^2(k-k_0)^2} \sqrt{2\pi a^2}$$

$$= \chi_0\, e^{-\frac{1}{2}\frac{(z+L_1)^2}{a^2}}\, e^{ik_0(z+L_1)}$$

Actually, I should have written:

$$\chi(t=0, z) = \chi_0 \int \frac{dk}{2\pi}\, e^{ikL_1} e^{-\frac{1}{2}a^2(k-k_0)^2} \sqrt{2\pi a^2}\, [e^{ikz} - e^{-ikz}]$$

but this is

$$= \chi_0 \left\{ e^{-\frac{1}{2}\frac{(z+L_1)^2}{a^2}} e^{ik_0(z+L_1)} - e^{-\frac{1}{2}\frac{(L_1-z)^2}{a^2}} e^{ik_0(L_1-z)} \right\}$$

and, if $L_1 \gg a$ and $z < 0$, this second term is completely negligible.

Now add time evolution. We have:

$$\chi(t, z) = \chi_0 \int \frac{dk}{2\pi} \sqrt{2\pi a^2}\, e^{-\frac{1}{2}a^2(k-k_0)^2} e^{ikL_1} \left[e^{ikz - i\omega t} - e^{-ikz - i\omega t} \right]$$

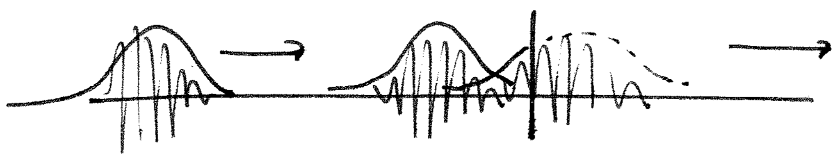
write $\omega = ck$ and perform the integral. We find

$$\chi(t,z) = \chi_0 \left\{ e^{-\frac{1}{2}\frac{(L_1+z-ct)^2}{a^2}} e^{ik_0(L_1+z-ct)} - e^{-\frac{1}{2}\frac{(L_1-z-ct)^2}{a^2}} e^{ik_0(L_1-z-ct)} \right\}$$

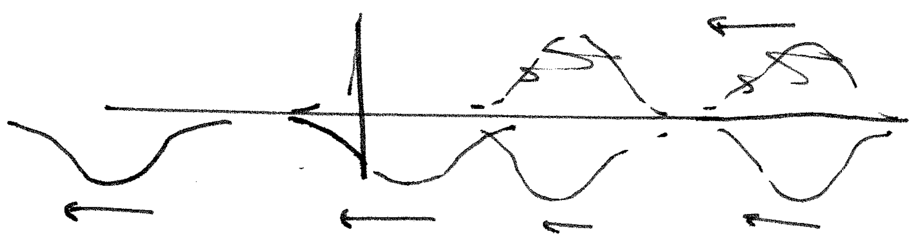
The first term is a Gaussian centered at

$$z = -L_1 + ct$$

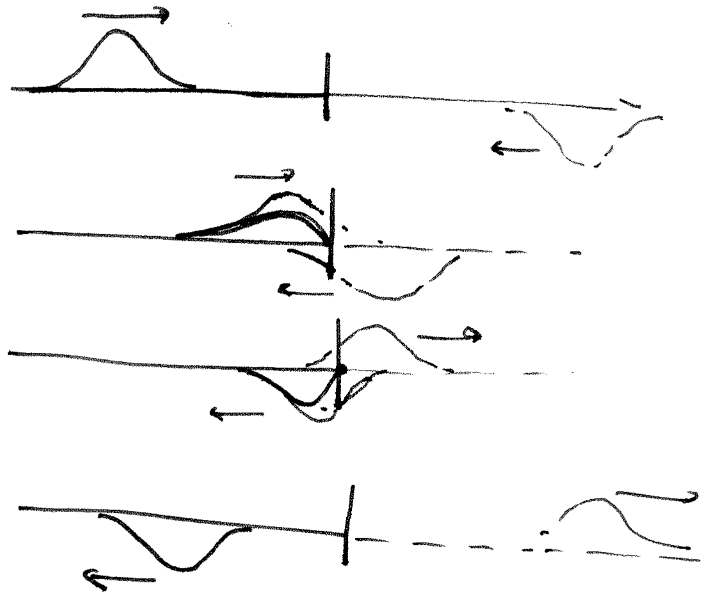
that eventually disappears into the wall at $z=0$



The second term is an inverted Gaussian image pulse that emerges from behind the wall and moves to the left



with center at $z = L_1 - ct$. The superposition of these pulses gives a motion:

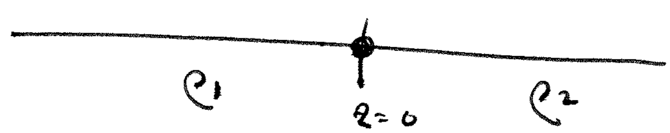


The superposition describes a wave packet that starts at $z = -L_1$, takes a time $T = L_1/c$ to reach $z=0$, then reflects from the wall at $z=0$ and moves back to the left. When the wave returns to $z = -L_1$, the total phase accumulation of the wave is

$$2 k_0 L_1 + \pi$$

where π counts the inversion of the reflected packet.

Here is a somewhat hard problem. Imagine that we have two strings of different density ρ, joined together at $z=0$



If we start a wavepacket in the region to the far left, what happens?

First, we should understand the boundary condition at $z=0$. This should be

$$\chi(z=0-\epsilon) = \chi(z=0+\epsilon) \quad \text{ie } \chi(z) \text{ continuous at } z=0$$
$$\text{and} \quad \partial\chi/\partial z(z=0-\epsilon) = \partial\chi/\partial z(z=0+\epsilon)$$

The string with $z<0$ has energy

$$E_1 = \int_{-\infty}^{0} dz \left[\tfrac{1}{2}\rho_1\left(\frac{\partial\chi}{\partial t}\right)^2 + \tfrac{1}{2}\kappa\left(\frac{\partial\chi}{\partial z}\right)^2\right] \qquad c_1 = \sqrt{\frac{\kappa}{\rho_1}}$$

The string with $z>0$ has energy

$$E_2 = \int_{0}^{\infty} dz \left[\tfrac{1}{2}\rho_2\left(\frac{\partial\chi}{\partial t}\right)^2 + \tfrac{1}{2}\kappa\left(\frac{\partial\chi}{\partial z}\right)^2\right] \qquad c_2 = \sqrt{\frac{\kappa}{\rho_2}}$$

If the joined strings are at equilibrium, they must have the same tension κ. It is not difficult to work out that the energy current is

$$\vec{j}_\mathcal{E} = \hat{z}\,\kappa\,\frac{\partial\chi}{\partial t}\frac{\partial\chi}{\partial z}$$

in particular, the expression for $j_\mathcal{E}$ is the same on both sides, so

$$\frac{d}{dt}(E_1+E_2) = \int_{-\infty}^{0} dz\,\frac{\partial\mathcal{E}_1}{\partial t} + \int_{0}^{\infty} dz\,\frac{\partial\mathcal{E}_2}{\partial t}$$

$$= \int_{-\infty}^{0} dz\,\frac{\partial}{\partial z} j_\mathcal{E} + \int_{0}^{\infty} dz\,\frac{\partial}{\partial z} j_\mathcal{E}$$

$$= j_\mathcal{E}(0) - j_\mathcal{E}(0)$$

$$= 0$$

Now let's find a wave solution satisfying these boundary conditions which can be interpreted as a wave coming in from the left and scattering on the boundary. The initial wave should have

$$\text{wavenumber } k \qquad \text{frequency } \omega = c_1 k$$

This wave wiggles the boundary point at the frequency ω. So, the wave on the other side should have the same ω, but then a different k.

$$\omega = c_1 k = c_2 k_2 \quad \rightarrow \quad k_2 = \frac{c_1}{c_2} k$$

Let's try, then, to find a solution of the form:

$$\chi_k(t,z) = \begin{cases} \mathrm{Re}\{\chi_0 (e^{ikz} + R e^{-ikz}) e^{-i\omega t}\} & z<0 \\ \mathrm{Re}\{\chi_0 T e^{ik_2 z} e^{-i\omega t}\} & z>0 \end{cases}$$

R = "reflection coefficient"
T = "transmission coefficient"

Continuity of χ gives

$$\chi_0 (1 + R) = \chi_0 T$$

Continuity of $\partial\chi/\partial z$ gives

$$\chi_0 (ik - ikR) = \chi_0 i k_2 T$$

Now we can solve for R and T:

$$(1+R) = T \qquad 1-R = \frac{k_2}{k} T = \frac{c_1}{c_2} T$$

$$R = \frac{c_2-c_1}{c_2+c_1} \qquad T = \frac{2c_2}{c_1+c_2}$$

Note that:

① $R > 0$ if $c_2 > c_1$ $\quad \rho_2 < \rho_1$

$R < 0$ if $c_2 < c_1$ $\quad \rho_2 > \rho_1$

In the extreme limit $\rho_2 \to 0$, $c_2 \to 0$ and we have the case of simple reflection: $R = -1 \quad T = 0$

② R and T obey the relation

$$1 = R^2 + \frac{c_1}{c_2} T^2$$

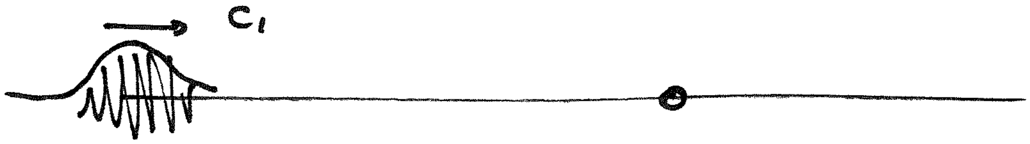
for all conditions.

Now, what is the physical interpretation of this solution? Let's take linear combinations of these solutions to form a wavepacket centered at $z = -L$, as before

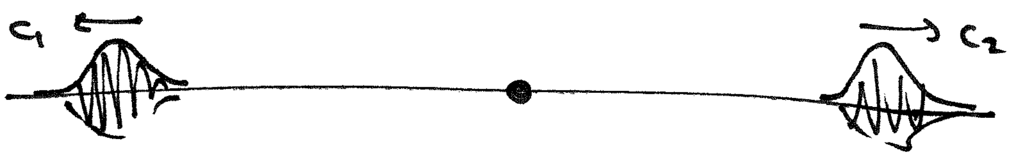
$$\int \frac{dk}{2\pi} \sqrt{2\pi a^2}\, e^{-\frac{1}{2}a^2(k-k_0)} \chi_k(t,z)\, e^{ikL}$$

At $t=0$, this is again a very good approximation to the wavepacket

$$\chi_0 \, e^{-\frac{1}{2}\frac{(L+z)^2}{a^2}} \, e^{ik_0(z+L)}$$



After a time $t = L/c$, the wavepacket will reach the join and pass through. Long after this time, the solution will consist of two wavepackets



The reflected packet has the form computed above, times the "reflection coefficient" R:

$$R \cdot \chi_0 \, e^{-\frac{1}{2}\frac{(L-z-ct)^2}{a^2}} \, e^{ik_0(L-z-ct)}$$

For the transmitted packet, we need to carefully evaluate the integral

$$\int \frac{dk}{2\pi} \sqrt{2\pi a^2} \, e^{-\frac{a^2}{2}(k-k_0)^2} \, e^{ikL} \, e^{ik_2 z - i\omega t}$$

with $k_2 = \frac{c_1}{c_2} k$.

This gives:

$$e^{-\frac{1}{2}\frac{1}{a^2}\left(L + \frac{c_1}{c_2}(z-c_2t)\right)^2} e^{ik_0\left(L + \frac{c_1}{c_2}(z-c_2t)\right)}$$

$$= e^{-\frac{1}{2}\frac{(L_2+z-c_2t)^2}{a_2^2}} e^{ik_{02}(L_2+z-c_2t)}$$

where $L_2 = \frac{c_2}{c_1}L \qquad a_2 = \frac{c_2}{c_1}a \qquad k_{02} = \frac{c_1}{c_2}k_0$

so the transmitted packet is

$$T \cdot \chi_0 \, e^{-\frac{1}{2}\frac{(L_2+z-c_2t)^2}{a_2^2}} e^{ik_{02}(L_2+z-c_2t)}$$

which has the correct time delay $t = \frac{L}{c} = \frac{L_2}{c_2}$, the correct wave number k_{02}; and a contracted or expanded size (for $c_2<c_1$, $c_2>c_1$, resp.) a_2.

Let's now check energy conservation. The original energy in the packet was.

$$\frac{1}{2}\rho_1\omega^2|\chi_0|^2 \cdot \sqrt{\pi}\,a$$

The energy in the packet moving to the left is

$$\frac{1}{2}\rho_1\omega^2|\chi_0|^2\,|R|^2\,\sqrt{\pi}\,a$$

The energy in the packet moving to the right is

$$\tfrac{1}{2}\rho_2 \omega^2 |\chi_0|^2 |T|^2 \sqrt{\pi}\, a_2$$

$$= \tfrac{1}{2}\rho_1 \omega^2 |\chi_0|^2 |T|^2 \cdot \sqrt{\pi}\, a \cdot \left(\frac{\rho_2}{\rho_1} \frac{a_2}{a} \right)$$

The factor in parentheses is

$$\frac{c_1^2}{c_2^2} \cdot \frac{c_2}{c_1} = \frac{c_1}{c_2}$$

Thus, energy conservation predicts

$$1 = |R|^2 + \frac{c_1}{c_2} |T|^2$$

which is precisely the relation that we found in our explicit computation.