Physics 121

Intermediate Electricity and Magnetism

Syllabus:

Magnetic induction

Response of linear systems

Maxwell's equations

Electromagnetic waves

Energy and momentum of the electromagnetic field

Waves in media: transmission, reflection, refraction

Wave guides

Theory of relativity

References

textbook:

Griffiths, Introduction to Electrodynamics

also:

Heald and Marion, Classical Electromagnetic Radiation

and

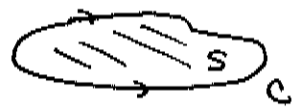
Feynman, Leighton, and Sands,
The Feynman Lectures in Physics, vol 2

note: Heald + Marion use CGS or Gaussian units.
If you are confused, check the conversion tables in H+M, Appendix D + E. I will continue to use SI units in the lecture notes.

Jan. 10

Magnetic Induction

At the end of Physics 120, we justified Faraday's law, the statement that a change in magnetic flux leads to an electric force:

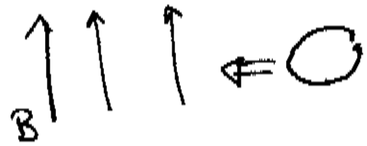


$$\frac{d}{dt}\Phi_S = -\int_C d\vec{\ell}\cdot\vec{E}$$

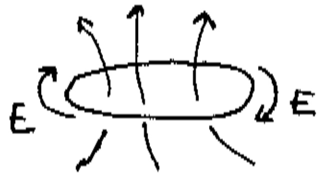
or, in differential form

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

This law follows from Faraday's intuition — and experimental verification — of the idea that, if moving a conducting loop into a magnetic field leads to a voltage induced on the loop

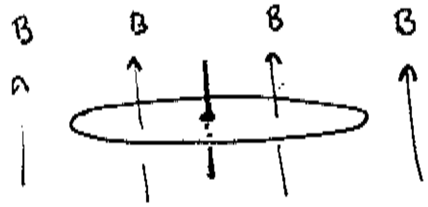


any other method of changing the magnetic flux through the loop should lead to a similar induced voltage. The sign of the induced voltage is such as to drive a current that opposes the change in flux (Lenz's law).

$\frac{d\Phi}{dt} > 0$  $\oint d\vec{l}\cdot\vec{E} < 0$

Faraday's law is our first <u>time</u> depen<u>dent</u> field equation, and it is amazing that it lin<u>k</u>s electric and magnetic fields. When we pursue this idea to its logical conclusion, we will end up with a complete dynamical theory of electric and magnetic fields — Maxwell's equations. But, that is coming a little later. First, I would like to discuss some practical details of magnetic induction.

Let's discuss this first in a pretty toy example given by Griffiths. Consider a circular loop of radius a, with charge density ρ (C/m), mounted so that it can rotate on its axis.



$\vec{B} = B\hat{z}$

and imagine that a magnetic field $\vec{B}$ threads the loop. Now turn off $\vec{B}$. What happens?

First, there is a $\frac{d\Phi}{dt} = \pi a^2 \frac{dB}{dt}$

This in turn leads to an $\vec{E}$ around the loop: $\oint d\vec{l}\cdot\vec{E} = -\pi a^2 \frac{dB}{dt}$

$$2\pi a E_\phi = -\pi a^2 \frac{dB}{dt}$$

$$E_\phi = -\frac{a}{2}\frac{dB}{dt}$$

This in turn leads to a torque on the loop:

$$\vec{\tau} = \oint \vec{r} \times d\vec{F} = \oint a\hat{r} \times \rho E_\phi a \, d\phi \, \hat{\phi}$$

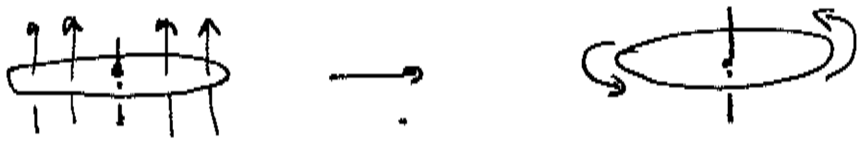
$$= 2\pi\rho a^2 E_\phi \hat{z}$$

$$\vec{\tau} = -\pi a^3 \rho \frac{dB}{dt}$$

since $\vec{\tau} = \frac{d\vec{L}}{dt}$, the loop receives angular momentum

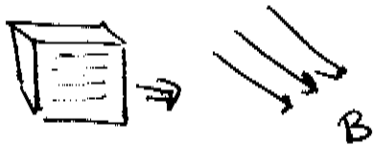
$$\vec{L} = -\pi a^3 \rho \, \Delta B \, \hat{z}$$

If we turn off the field, the loop spins up

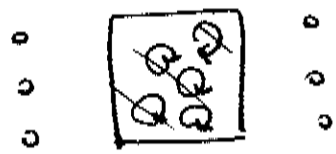


It seems that a $\vec{B}$ field carries angular momentum, which is transferred to the loop.

A related, more practical, phenomenon, is that of eddy currents. If we move a sheet of conductor into a magnetic field

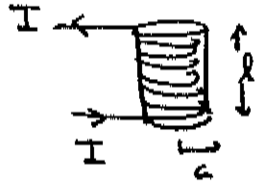


each small area of the surface feels a change in flux and an associated $\oint d\vec{l}\cdot\vec{E}$. So, current flows in small loops



opposing the magnetic field and causing the energy of motion to be dissipated.

Next, consider a solenoid with radius a and n turns/m and length l



We derived last term that this solenoid contains a magnetic field

$$B = \mu_0 I n$$

so the flux through each turn is $\Phi/\text{turn} = \mu_0 I n \pi a^2$

and the total flux is

$$\Phi = \mu_0 I n \cdot \pi a^2 n \cdot l$$

Let I change, but slowly enough that the formula for B remains approximately correct. Then a voltage is induced across the coil

$$V = -\frac{d\Phi}{dt} = -\mu_0 \pi a^2 n^2 l \frac{dI}{dt}$$

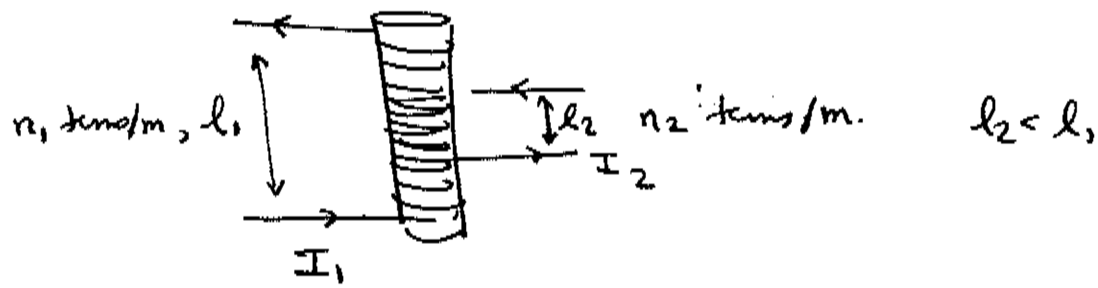
by Lenz's law, the direction of the voltage opposes the change in I. This relation is written symbolically as

$$V = -L \frac{dI}{dt}$$

L is called the inductance (or "self-inductance") of the coil. L is measured in Henries

$$1 \text{ henry } (1\,H) = 1\ V \cdot \frac{sec}{A}$$

Two circuits can interact through magnetic induction. Here is a simple example: Imagine two solenoids wrapped on the same (nonmagnetic) cylinder.



If a current I_1 is flowing, the solenoid contains magnetic field

$$B = \mu_0 I_1 n_1$$

the flux through the second circuit is

$$\Phi_2 = \mu_0 I_1 n_1 \cdot \pi a^2 n_2 l_2$$

If I_1 changes in time, a voltage is induced in circuit 2.

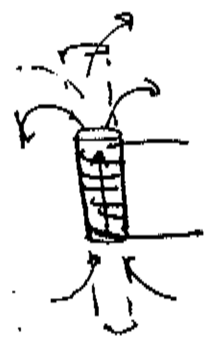
$$V_2 = -\mu_0 \pi a^2 n_1 n_2 l_2 \frac{dI_1}{dt}$$

symbolically

$$V_2 = -M_{21} \frac{dI_1}{dt}$$

M_{21} is called the ~~mutual~~ mutual inductance. If $I_1(t)$ is an alternating current ($I_1(t) = I_{10} \cos\omega t$) we can engineer the circuit so that the voltage is either greater or less in the circuit 2. This is how an AC transformer operates.

Let's now compute the effect of circuit 2 on circuit 1. Let me first make the drastic approximation that the field of the shorter solenoid 2 is rapidly decreasing outside its length:



Then the field in the solenoid due to I_2 is $B = \mu_0 I_2 n_2$ and the flux through circuit 1 is

$$\Phi_1 \cong \mu_0 I_2 n_2 \cdot \pi a^2 \cdot n_1 \cdot \underline{l_2}$$

so

$$V_1 \cong -\mu_0 \pi a^2 n_1 n_2 l_2 \frac{dI_2}{dt}$$

We find

$$V_2 = -M_{21} \frac{dI_1}{dt} \qquad V_1 = -M_{12} \frac{dI_2}{dt}$$

with $M_{21} \cong M_{12} \cong \mu_0 \pi a^2 n_1 n_2 l_2$

Actually, the equality of the mutual inductances $M_{21} = M_{12}$ is exact. To see this, let's compute the exact flux of $\vec{B}$ through circuit 2 due to circuit 1. The easiest way to do this is to compute the $\vec{A}$ field due to circuit 1.

$$\vec{A}_1(\vec{x}) = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} d\vec{l}_y \frac{1}{|\vec{x}-\vec{y}|}$$

[Recall that taking $\vec{\nabla} \times \vec{A}$ gives the Biot-Savart law:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l} \times (\vec{x}-\vec{y})}{|x-y|^3} \quad .]$$

The flux of B through circuit 2 can be computed as

$$\Phi_2 = \int d^2x \, \hat{n} \cdot \vec{B}_1 = \oint_{C_2} d\vec{l} \cdot \vec{A}_1$$

$$= \oint_{C_2} d\vec{l}_x \cdot \frac{\mu_0}{4\pi} I_1 \oint_{C_2} d\vec{l}_y \frac{1}{|\vec{x}-\vec{y}|}$$

so

$$\frac{d\Phi_2}{dt} = -V_2 = M_{21} \frac{dI_1}{dt}$$

where

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{C_1}\oint_{C_2} d\vec{l}_x \cdot d\vec{l}_y \frac{1}{|\vec{x}-\vec{y}|}$$

This expression is perfectly symmetrical between the two loops, so

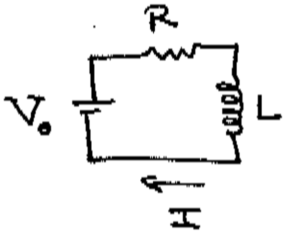
$$M_{12} = M_{21}.$$

We can idealize a coil or other circuit with the relation

$$V = -L\frac{dI}{dt}$$

as an ideal circuit element, an "inductor". We can combine inductors with resistors and capacitors in circuits. (Actually, every circuit element has a small inductance, resistance, capacitance.) Let's study this in simple situations.

Consider first the circuit



The total potential drop around the circuit is

$$V_0 = IR + L\frac{dI}{dt}$$

The steady-state current is $I = \frac{V_0}{R}$. However, if we start from $I = 0$ (open circuit) and close the circuit, we have to solve this differential equation with $I(t) = 0$ at $t = 0$. The solution is

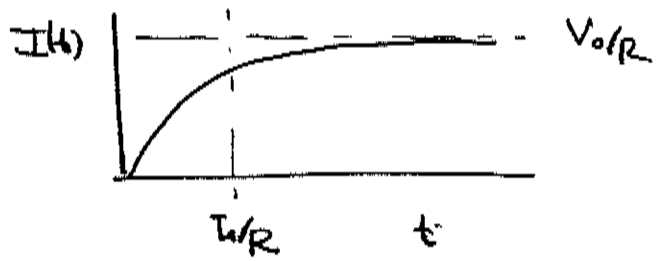
$$I(t) = \frac{V_0}{R}\left[1 - e^{-(R/L)t}\right]$$

the current can not suddenly jump to V_0/R because changes in I are opposed by the inductor. The characteristic time to change I is

$$\tau = L/R \qquad \text{units: } \frac{H}{\Omega} = \frac{V\,sec/A}{V/A} = sec.$$

For some typical values $L \sim 10^{-6}\,H$ $R \sim 10^{3}\,\Omega$ $\tau \sim 10^{-9}\,sec.$

The form of $I(t)$ for this situation is:



larger $L \rightarrow$ larger τ

Circuits with both inductance and capacitance have more interesting behavior, which we will discuss next time.