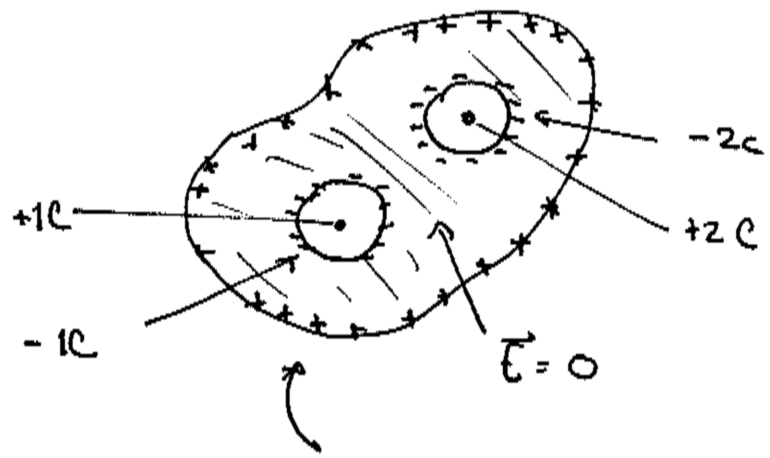
Physics 120 – Final Exam

Solutions

1.) Short Answer:

a.) Neutralizing charge covers the wall of each spherical cavity. The force on each fixed charge is zero.



the charge on the outside surface is + 8C.

b.) i.) the leading multipole moment is a monopole $Q = 2$

$$\phi \sim \frac{1}{4\pi\epsilon_0} \frac{(2C)}{r}$$

(ii) the leading multipole moment is a dipole

$$\vec{d} = (+2a + 1a)(1, 0, 0)$$

so $\vec{d} = 3a\,(1,0,0)$

$$\phi \sim \frac{1}{4\pi\epsilon_0}\,\frac{\vec{d}\cdot\hat{r}}{r^2} = \frac{3a}{4\pi\epsilon_0}\,\frac{x}{r^3}$$

(iii) the leading multipole moment is a quadrupole

$$Q_{ij} = \begin{pmatrix} 2(-1)\cdot(3a^2-a^2) & & \\ & 2(-1)(-a^2) & \\ & & 2(-1)(-a^2) \end{pmatrix}$$

$$= \begin{pmatrix} -4a^2 & & \\ & 2a^2 & \\ & & 2a^2 \end{pmatrix}$$

$$\phi \sim \frac{1}{4\pi\epsilon_0}\,\frac{1}{2r^3}\,Q_{ij}\,\hat{r}_i\,\hat{r}_j = \frac{a^2}{4\pi\epsilon_0}\,\frac{(-2x^2+y^2+z^2)}{r^5}$$

$$= \frac{-a^2(3\cos^2\theta-1)}{4\pi\epsilon_0 r^3}$$ where θ is measured from the $\hat{x}$ axis

(c) In the solenoids, the magnetic field is

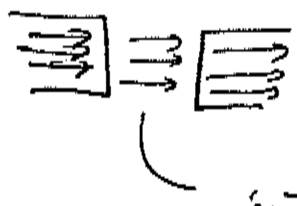
$$\mu_0 H = \mu_0 n I = (4\pi\times10^{-7}\,\mathrm{N/A^2})(10^4/\mathrm{m})(1\,\mathrm{A})$$

$$= 4\pi\times10^{-3}\,\mathrm{T} = 4\pi\times10\ \text{gauss}$$

$$= 130\ \text{gauss}.$$

$$B = 13{,}000\ \text{gauss}. = 1.3\,\mathrm{T}$$

Since $B_\perp$ is constant across the surface gap



in the gap, $B = 1.3\ T$

(d.) $F = QvB = \frac{mv^2}{r}$

so $r = \frac{mv}{QB}$

v is given by $\frac{1}{2}mv^2 = E = 1\text{ MeV}$

$$\frac{\frac{1}{2}mv^2}{mc^2} = \frac{1\text{ MeV}}{938\text{ MeV}} \Rightarrow v = \left(\frac{2}{938}\right)^{1/2} c$$

$$= 4.6\times10^{-2} \cdot c$$

$$r = \frac{mv}{eB} = \frac{mc^2\ v/c \cdot 1/c}{e\cdot(10^{-4}\,T)}$$

$$= \frac{938\times10^{6}\ \cancel{e}\cdot(1\text{ Volt})\ (4.6\times10^{-2})}{\cancel{e}\ 10^{-4}\,T \cdot 3\times10^{8}\,m/sec}$$

$$= 1.4\times10^{3}\ \frac{V\cdot sec}{T\cdot m}$$

the units are $\frac{kg\,m^2/sec^2 C \cdot sec}{(kg\,m^2/sec^2/C/sec)\cdot m}$

$= m$

$= 1.4$ km

(e.) The capacitance is

$$C = \epsilon_0 \frac{A}{d} = \left(8.85 \times 10^{-12}\, C^2/Nm^2\right) \cdot \frac{(3\times10^{-3})^2}{10^{-3}}\, m$$

$$= 8.9 \times 10^{-12} \cdot 9 \times 10^{-3} \quad F$$

$$= 8 \times 10^{-14}\, F$$

$$\text{energy} = \frac{1}{2} C V^2 = \frac{1}{2} \cdot 8\times10^{-14} \cdot (120)^2 = 5.8\times10^{-10}\, J$$

$$\text{time to discharge} = RC = 8\times10^{-14}\, F \cdot 10^{4}\, \Omega$$

$$= 8\times10^{-10}\, \text{sec}$$

2.) (a.) The force on site (i,j) from site $(i-1,j)$ is

$$\vec{F} = -k(\vec{x}_{(i,j)} - x_{(i-1,j)})$$

in the $\hat{z}$ direction

$$F_z = -k(h(i,j) - h(i-1,j)) \qquad \text{where } h(i,j) = \text{height} = z(i,j)$$

The equation of the balance of forces is

$$0 = [h(i,j) - h(i-1,j)] + [h(i,j) - h(i+1,j)] + [h(i,j) - h(i,j-1)] + [h(i,j) - h(i,j+1)]$$

$$0 = 4h(i,j) - h(i-1,j) - h(i+1,j) - h(i,j-1) - h(i,j+1)$$

$$0 = [h(i+1,j) - 2h(i,j) - h(i-1,j)] + [h(i,j+1) - 2h(i,j) + h(i,j-1)]$$

which is a discrete version of

$$0 = \frac{\partial^2}{\partial x^2}h + \frac{\partial^2}{\partial y^2}h$$

(b.) We know from 2-D electrostatics that

$$\phi = \text{const} \cdot \log r$$

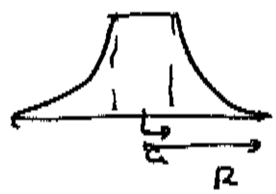
solves the Laplace equation in 2-D. From this function, we can build the solution:

$$\phi = A \log R/r$$

this $= 0$ for $r = R$, for $r = a$ $\quad \phi = A \log R/a$

so

$$\phi = T_1 \frac{\log R/r}{\log R/a}$$



solves all necessary conditions

(c.) Let $\vec{d} = (d, 0)$. Then

$$\phi = A \left(\log |\vec{d} + \vec{x}| - \log |\vec{d} - \vec{x}| \right)$$

$$= \frac{A}{2} \log \frac{(d+x)^2 + y^2}{(d-x)^2 + y^2}$$

solves the Laplace eq. and satisfies $\phi = 0$ for $x = 0$

at $|\vec{d} - \vec{x}| = a$

$$\phi \cong \frac{A}{2} \log \frac{(2d)^2}{a^2} \qquad \text{for } a \ll d$$

so

$$\phi = T_1 \frac{\log \left[\frac{(d+x)^2 + y^2}{(d-x)^2 + y^2} \right]}{\log \frac{4d^2}{a^2}}$$

solves the complete problem for $a \ll d$.

3.) (a) Since the problem has cylindrical symmetry

outside

$$\phi = \sum_\ell \left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

inside

$$\phi = \sum_\ell C_\ell f_\ell(r)\, P_\ell(\cos\theta)$$

where f_ℓ is an appropriate radial function. As $r \to \infty$ ϕ must become the potential of a constant field. So

$$\phi = -E_0 r P_1(\cos\theta) + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Fixing the boundary condition at $r=a$ will fix the coefficients B_ℓ and C_ℓ in terms of E_0. Since the various modes ℓ are independent, we can find a solution with all $B_\ell, C_\ell = 0$ except for $\ell = 1$. So

$$\phi = \left(-E_0 r + \frac{B_1}{r^2}\right) P_1(\cos\theta) \qquad \text{outside}$$

$$\phi = C_1 f_1(r)\, P_1(\cos\theta) \qquad \text{inside}$$

(b) The equation

$$\vec{\nabla}\cdot\vec{D} = 0$$ inside becomes, in spherical coordinates

$$\vec{\nabla}\cdot\ \varepsilon(r)\,\vec{E}(r) = -\vec{\nabla}\left(\varepsilon(r)\,\vec{\nabla}\phi\right) = 0$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}\ \varepsilon(r)\, r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right)\phi(r,\theta) = 0$$

$$\phi = f_\ell(r)\, P_\ell(\cos\theta)$$

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\ \underbrace{\varepsilon(r)\, r^2}_{p}\ \frac{\partial}{\partial r} + \frac{\varepsilon(r)}{r^2}\ell(\ell+1)\right] f_\ell(r) = 0$$

put $\varepsilon(r) = a^2/r^2$

$$\left[-\frac{a^2}{r^2}\frac{\partial^2}{\partial r^2} + \frac{a^2}{r^2}\ \frac{\ell(\ell+1)}{r^2}\right] f_\ell(r) = 0$$

$$\frac{\partial^2}{\partial r^2} f_\ell(r) = \frac{\ell(\ell+1)}{r^2} f_\ell(r)$$

so $$f_\ell(r) = A_\ell\, r^{\ell+1} + B_\ell \frac{1}{r^\ell}$$

the solution regular at the origin is $$f_\ell(r) = A_\ell\, r^{\ell+1}$$

$$f_1(r) = A_1 r^2$$

(c) The irregular solution gives $E \sim \frac{1}{r^2}$ as $r \to 0$

then

$$\frac{1}{2} \int d^3x\, \vec{E} \cdot \vec{D}$$

$$\sim \frac{1}{2} \int dr\, r^2\, \varepsilon(r) \left(\frac{1}{r^2}\right)^2 \sim \int dr\, \frac{1}{r^4}$$

is infinite!

so this solution is unacceptable.

(d) so $\phi = \left(-E_0 r + \frac{B_1}{r^2}\right) P_1(\cos\theta)$ outside

$\phi = C_1 r^2\, P_1(\cos\theta)$ inside.

$\varepsilon(r) = 1$ at $r = a$. So $E_\parallel$ and $E_\perp$ are continuous at $r = a$.

$E_\parallel : \frac{1}{r}\frac{\partial}{\partial\theta}$ $\quad -\frac{E_0 a}{a} + \frac{B_1}{a^3} = \frac{C_1 a^2}{a}$

$E_\perp : \frac{\partial}{\partial r}$ $\quad -E_0 - 2\frac{B_1}{a^3} = 2C_1 a$

$$-E_0 + 4\frac{B_1}{a^3} = 0 \qquad B_1 = \frac{E_0 a^3}{4}$$

$$\Rightarrow \quad C_1 = -\frac{3}{4} E_0 \frac{1}{a}$$

so outside

$$\phi = \left(-E_o r + \frac{1}{4}\frac{E_o a^3}{r^2}\right)\cos\Theta$$

inside

$$\phi = -\frac{3}{4}E_0\frac{r^2}{a}\cos\Theta$$

$$\vec{E} = -\hat{r}\frac{\partial}{\partial r}\phi + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\Theta}\phi$$

$$= \hat{r}\left(-\frac{3}{4}E_0\frac{2r}{a}\cos\Theta\right) + \hat{\Theta}\left(-\frac{3}{4}E_0\frac{r}{a}\right)(-\sin\Theta)$$

$$\vec{E} = +\frac{3}{4}E_0\frac{r}{a}\left(2\cos\Theta, -\sin\Theta\right)$$

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E} = \frac{3}{4}(\epsilon-\epsilon_0)E_0\frac{r}{a}\left(2\cos\Theta, -\sin\Theta\right)$$

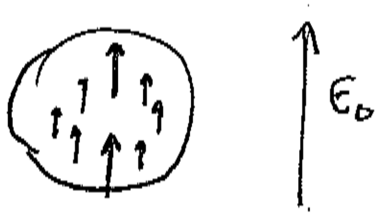
$$(\cos\Theta\,\hat{r} + (-\sin\Theta)\hat{\Theta}) = \hat{z}$$

so

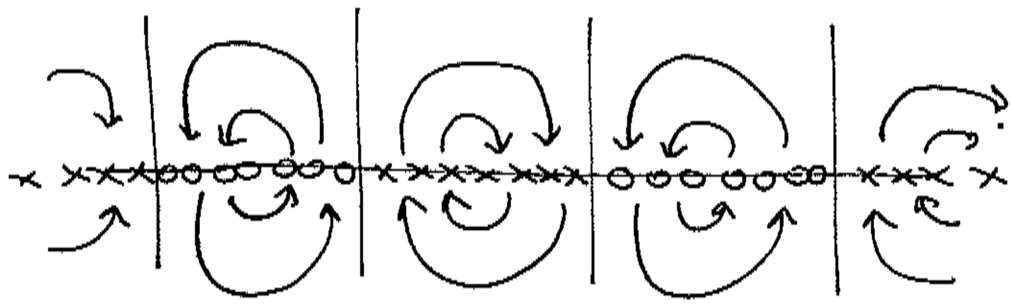
$$\vec{P} = \frac{3}{4}(\epsilon-\epsilon_0)E_0\frac{r}{a}\left(\hat{z} + \cos\Theta\,\hat{r}\right)$$

$$= \frac{3}{4}\left(\frac{a^2}{r^2}-1\right)\epsilon_0\frac{r}{a}E_0\left(\hat{z} + \cos\theta\,\hat{r}\right)$$

$$= \frac{3}{4}\left(\frac{a}{r}-\frac{r}{a}\right)\epsilon_0 E_0\left(\hat{z} + \cos\Theta\,\hat{r}\right)$$

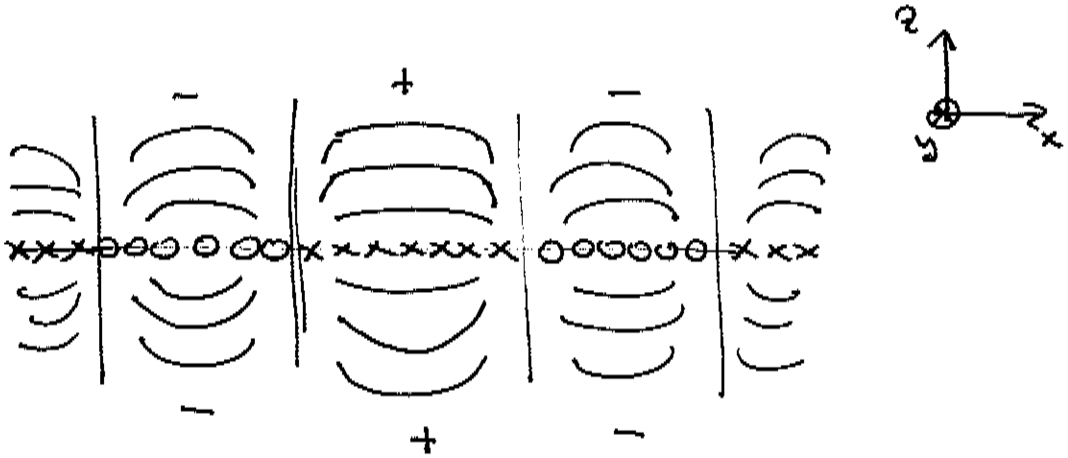


4.) (a) Qualitatively, the field should look like



$\vec{B}$ is maximal at the centers of the bands and zero on the ~~indicated~~ lines.

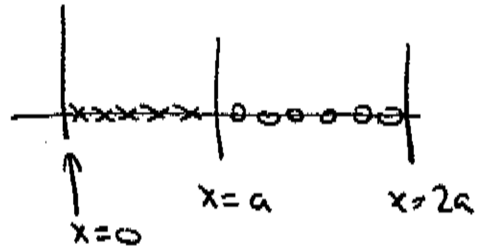
(b) $\vec{A}$ obeys a Poisson eq. whose source is the current. So there is a gauge where only A^y is nonzero. $\hat{y}$ points into the paper. Then A^y has contour lines:



A^y is zero on the vertical lines

(c.) We can solve for A^y as a Fourier series. The symmetry and periodicity implies.

$$A^y(x,z) = \sum_{n=1}^{\infty} f_n(z) \sin \frac{\pi n x}{a}$$



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) A^y(x,z) = 0 \Rightarrow$$

$$\frac{\partial^2}{\partial z^2} f_n(z) - \left(\frac{\pi n}{a}\right)^2 f_n(z) = 0$$

so $$f_n(z) = e^{-\frac{\pi n z}{a}} \text{ or } e^{+\frac{\pi n z}{a}}$$

if $f_n \to 0$ as $|z| \to \infty$

$$A^y(x,z) = \begin{cases} \sum_{n=1}^{\infty} c_n \, e^{-\frac{\pi n z}{a}} \sin \frac{\pi n x}{a} & z>0 \\ \sum_{n=1}^{\infty} c_n \, e^{+\frac{\pi n z}{a}} \sin \frac{\pi n x}{a} & z<0 \end{cases}$$

the discontinuity in $\frac{\partial}{\partial z} A^y$ should be the "surface charge" $\mu_0 \mathcal{J}$

$$\sum_{n=1}^{\infty} c_n \frac{2\pi n}{a} \sin\frac{\pi n x}{a} = \mu_0 n I \cdot \begin{cases} 1 & 0<x<a \\ -1 & a<x<2a \end{cases}$$

overlap w. $\sin\frac{\pi k x}{a}$

$$\int_0^{2a} dx \, \sin\frac{\pi k x}{a} \sum_{n=1}^{\infty} c_n \frac{2\pi n}{a} \sin\frac{\pi n x}{a}$$

$$= c_k \frac{2\pi k}{a} \cdot a$$

$$= \mu_0 n I \underbrace{\int_0^{2a} dx \, \sin\pi\frac{kx}{a} \cdot \begin{cases} 1 & x<a \\ -1 & x>a \end{cases}}_{\mathcal{J}}$$

$\mathcal{J} = 0$ for k even; for k odd

$$\mathcal{J} = 2\int_0^a dx \, \sin\frac{\pi k x}{a} = 2\frac{a}{\pi k}\int_0^{\pi k} dy \, \sin y$$

$$= \frac{4a}{\pi k}$$

So

$$c_k \, \frac{2\pi k}{\cdot} = \mu_0 n I \, \frac{4a}{\pi k}$$

$$c_k = \mu_0 n I \, \frac{2a}{\pi^2 k}$$

So

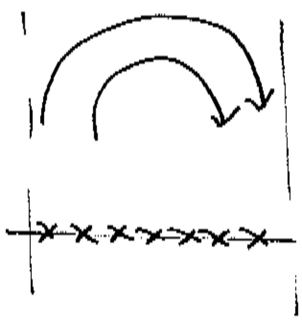
$$A^y = \sum_{k \text{ odd} = 1}^{\infty} \mu_0 n I \cdot \frac{2a}{\pi^2 k} \, e^{-\frac{\pi k z}{a}} \sin \frac{\pi k x}{a} \qquad (z > 0)$$

d.) the leading term is

$$A^y \sim \mu_0 n I \cdot \frac{2a}{\pi^2} \, e^{-\frac{\pi z}{a}} \sin \frac{\pi x}{a} \qquad z \to \infty$$

$$B^x = -\frac{\partial}{\partial z} A^y \sim \mu_0 n I \cdot \frac{2}{\pi} \, e^{-\frac{\pi z}{a}} \sin \frac{\pi x}{a}$$

$$B^z = \frac{\partial}{\partial x} A^y \sim \mu_0 n I \, \frac{2}{\pi} \, e^{-\frac{\pi z}{a}} \cos \frac{\pi x}{a}$$



as required