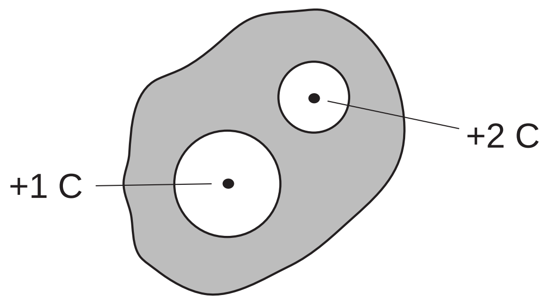
Physics 120 – Final Exam

(Monday, March 18)

There are five short-answer problems worth 8 points each and three long problems worth 20 points each, for a grand total of 100 points.

1. Short Answer; 8 points each

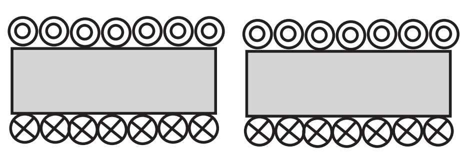
   (a) The conductor shown in this figure has residing on it a total charge of +5 C. Two spherical cavities carved out of the conductor contain fixed charges of +1 C and +2 C as shown, each charge placed at the center of the cavity. Find the force on each fixed charge and the total charge on each surface of the conductor.



   (b) Find the leading term in the electrostatic potential $\phi(\vec{r})$ as $r \to \infty$ for the following distributions of charge:

      i. +2 C at (a,0,0), -1 C at (0,0,0), +1 C at (-a,0,0).
      ii. +2 C at (a,0,0), -1 C at (0,0,0), -1 C at (-a,0,0).
      iii. -1 C at (a,0,0), +2 C at (0,0,0), -1 C at (-a,0,0).

   (c) Take two cylindrical pieces of a linear magnetic material, 10 cm long, 1 cm in diameter, with $\mu/\mu_0 = 100$. Wrap each with 1000 turns of a wire carrying 1 A. Place the solenoids end-to-end as shown in the figure. Compute the magnetic field in the small air gap between them.

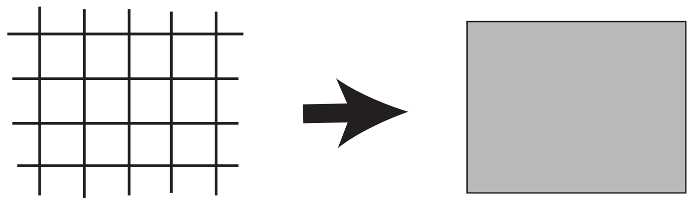


   (d) Cosmic rays (for definiteness, protons of energy 1 MeV) enter the earth's magnetic field (for definiteness, constant at 1 gauss). What is the radius of the helical path?

(e) You stick your finger in an electrical outlet. Model this as a temporarily isolated capacitor of area 3 mm $\times$ 3mm, a 1 mm gap, and 120 Volts across it. Take the air gap to have a resistance of $10^4\Omega$. Compute the energy released in the discharge, and the characteristic time it takes to discharge it. If this time is greater than 1/60 sec, you shouldn't feel anything.

2. (20 points) This problem studies the behavior of a sheet of rubber held under tension.

   (a) Model the rubber sheet as a lattice of springs, as shown in the figure. Each spring has spring constant k. Take the equilibrium plane of the rubber to be the $(\hat{x}, \hat{y})$ plane. Write the equation for the balance of forces in the $\hat{z}$ direction at each lattice point. You should find a discrete version of the Laplace equation.



   In the rest of this problem, use the fact that the height $h(x, y)$ of a rubber sheet obeys the Laplace equation.

   (b) A rubber sheet is nailed down ($h = 0$) on a circle of radius R. A cylindrical pole, of radius a, is pushed vertically into the center of the sheet sheet to a height L. Find $h(x, y)$. Use the cylindrical symmetry.

   (c) Now assume that the rubber sheet is large and that it is nailed down along a linear boundary at $x = 0$. The pole is pushed in at a distance d from this boundary. Find $h(x, y)$. You can assume that $a << d$.

3. (20 points) Consider a sphere of dielectric of radius a placed into a constant electric field $E_0\hat{z}$. The sphere is constructed to have a variable dielectric constant: $\epsilon/\epsilon_0 = a^2/r^2$. (Materials with spatially varying dielectric constant can be fabricated by building them from layers of plastic with different chemical composition. However, special materials in which $\vec{\nabla}\epsilon$ becomes infinite are available only on final exams.)

   (a) Write the general form of the solutions to Laplace's equation inside and outside the sphere. Show that the solution inside has the form of a a single Legendre polynomial multiplying a function $f(r)$.

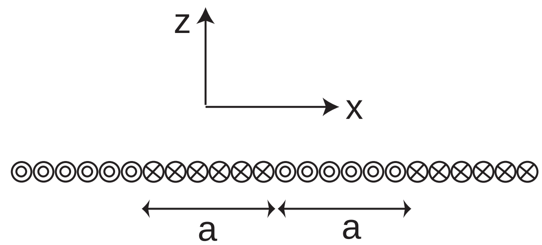
   (b) Write the differential equation for $f(r)$ and find its general solution.

   (c) Using the energy of electric fields in a dielectric, show that $f(r)$ must be 0 at $r = 0$.

   (d) Using all of the boundary conditions, find the electrostatic potential explicitly inside the sphere.

(e) Find the polarization $\vec{P}(x)$ in the material.

4. (20 points) Wires are set out on a board with a density of n wires/m. Each wire carries a current I. The direction of the current reverses with periodicity $2a$, as shown:



(a) Sketch the magnetic field set up by these wires. Show the approximate direction as a function of x and z. Are the maxima of $\vec{B}$ above the centers or the edges of the bands of wires?

(b) Sketch $\vec{A}(x,z)$ (in a convenient gauge). Where are the zeros of $\vec{A}(x,z)$?

(c) Solve for $\vec{A}(x,z)$ as an explicit series expansion. You can assume that $\vec{A}$ is symmetrical above and below the plane of the wires.

(d) Show that $\vec{A}$ decays exponentially as $|z| \to \infty$. Find the leading term for large positive z. Compute the associated leading term of the magnetic field.

Some useful quantities:

$$
\begin{aligned}
\epsilon_0 &= 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2} \\
\mu_0 &= 4\pi \times 10^{-7}\ \mathrm{N/A^2} \\
e &= 1.60 \times 10^{-19}\ \mathrm{C} \\
m_p &= 1.67 \times 10^{-27}\ \mathrm{kg} \\
m_p c^2 &= 938\ \mathrm{MeV} \\
c &= 3.00 \times 10^{8}\ \mathrm{m/sec}
\end{aligned} \tag{1}
$$

with 1 eV $= e \cdot 1$ Volt, 1 MeV $= 10^6$ eV. It follows from the above that 1 eV $= 1.6 \times 10^{-19}$ J.

$$
P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{2}
$$