Laplace's Equation in Spherical Coordinates

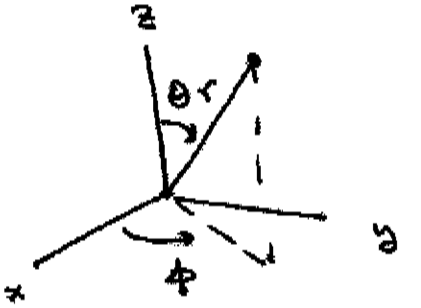
Oct 20

In the previous lecture, we saw that the method of solving Laplace's equation by Fourier series was an example of a more general method based on Sturm Liouville theory. Fourier series are well adapted to problems in which the boundary conditions are imposed on plane surfaces. But what if we want to solve Laplace's equation with boundary conditions on spherical surfaces. For example, what if we have a "spherical capacitor":

with $\phi = +V/2$ on the northern hemisphere and $\phi = -V/2$ on the southern hemisphere of a sphere of radius R. How can we find ϕ and $\vec{E}$ outside the sphere?

We might begin by setting up spherical coordinates and write the Laplacian operator $(-\nabla^2)$ in this coordinate system. In spherical coordinates, we parametrize

$$\vec{r} = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta)$$

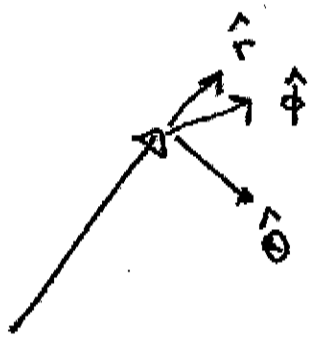


θ is the "polar" angle, ϕ is the "azimuthal" angle. In the vicinity of $\vec{x} = \vec{r}$, we can represent points by the x, y, z displacement from $\vec{r}$. But, alternatively, we can use the displacements along any other set of orthonormal coordinate axes. A convenient choice is:

$$\hat{r} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) = \frac{\partial\vec{r}}{\partial r}$$

$$\hat{\theta} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta) = \frac{1}{r}\frac{\partial\vec{r}}{\partial\theta}$$

$$\hat{\phi} = (-\sin\phi,\ \cos\phi,\ 0) = \frac{1}{r\sin\theta}\frac{\partial\vec{r}}{\partial\phi}$$



Notice that $(\hat{r}, \hat{\theta}, \hat{\phi})$ are not only mutually orthogonal, they form a right-handed coordinate system.

Notice that, if ds is an increment of path length (distance)

$$ds\ (\text{along } \hat r) = dr$$
$$ds\ (\text{along } \hat\theta) = r\,d\theta$$
$$ds\ (\text{along } \hat\phi) = r\sin\theta\,d\phi$$

Now $$\vec\nabla = \hat x \frac{\partial}{\partial x} + \hat y \frac{\partial}{\partial y} + \hat z \frac{\partial}{\partial z}$$

$$= \sum_i \hat\imath \frac{\partial}{\partial x_i}$$ where $\hat\imath$ are any three orthogonal axes.

and $\frac{\partial}{\partial x_i}$ is the derivative w.r. respect to path length along that axis.

then, using $\hat r\ \hat\theta\ \hat\phi$ as the axes.

$$\vec\nabla = \hat r \frac{\partial}{\partial r} + \hat\theta \frac{\partial}{r\,\partial\theta} + \frac{\hat\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Now that we have an expression for $\vec\nabla$ in spherical coordinates, we can easily work out expressions for div, curl., etc. We only need to take into account that $\hat r, \hat\theta, \hat\phi$ depend on r, θ, ϕ and must be differentiated correctly. To make this easier, I'll make a table of these derivatives:

$$\frac{\partial\hat r}{\partial r} = 0 \qquad \frac{\partial\hat r}{\partial\theta} = \hat\theta \qquad \frac{\partial\hat r}{\partial\phi} = \sin\theta\,\hat\phi$$

$$\frac{\partial\hat\theta}{\partial r} = 0 \qquad \frac{\partial\hat\theta}{\partial\theta} = -\hat r \qquad \frac{\partial\hat\theta}{\partial\phi} = \cos\theta\,\hat\phi$$

$$\frac{\partial\hat\phi}{\partial r} = 0 \qquad \frac{\partial\hat\phi}{\partial\theta} = 0 \qquad \frac{\partial\hat\phi}{\partial\phi} = \sin\theta\,\hat r + \cos\theta\,\hat\theta$$

note that $\hat{r}\left(\frac{\partial}{\partial\alpha}\hat{r}\right) = 0, \quad \hat{\theta}\cdot\frac{\partial}{\partial\alpha}\hat{\theta} = 0, \quad \hat{\phi}\cdot\frac{\partial}{\partial\alpha}\hat{\phi} = 0$

for $\alpha = r, \theta, \phi$. This must be true, because, $\hat{r}, \hat{\theta}, \hat{\phi}$ are <u>unit</u> vectors

$$\hat{r}\cdot\frac{\partial}{\partial\alpha}\hat{r} = \frac{1}{2}\frac{\partial}{\partial\alpha}(\hat{r}\cdot\hat{r}) = \frac{1}{2}\frac{\partial}{\partial\alpha}(1) = 0$$

Now compute

$$-\nabla^2 = -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

we may immediately drop terms with $\hat{r}\cdot\hat{\theta},\ \hat{r}\cdot\hat{\phi},\ \hat{\theta}\cdot\hat{\phi} = 0$

$\hat{r}\cdot\frac{\partial}{\partial\alpha}\hat{r},\ \hat{\theta}\cdot\frac{\partial}{\partial\alpha}\hat{\theta},\ \hat{\phi}\cdot\frac{\partial}{\partial\alpha}\hat{\phi} = 0$

what remains is:

$$= -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$-\hat{r}\left(\frac{\partial}{\partial r}\hat{\theta}\right)\frac{1}{r}\frac{\partial}{\partial\theta} - \hat{r}\left(\frac{\partial}{\partial r}\hat{\phi}\right)\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$-\hat{\theta}\,\frac{1}{r}\left(\frac{\partial}{\partial\theta}\hat{r}\right)\frac{\partial}{\partial r} - \hat{\theta}\,\frac{1}{r}\left(\frac{\partial}{\partial\theta}\hat{\phi}\right)\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$-\hat{\phi}\,\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\hat{r}\right)\frac{\partial}{\partial r} - \hat{\phi}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\hat{\theta}\right)\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$= -\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - 0 - 0$$

$$-\hat{\theta}\,\frac{1}{r}\,\hat{\theta}\,\frac{\partial}{\partial r} - 0 - \hat{\phi}\,\frac{1}{r\sin\theta}\,\sin\theta\,\hat{\phi}\,\frac{\partial}{\partial r} - \frac{\hat{\phi}}{r\sin\theta}\cos\theta\,\hat{\phi}\,\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$= -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

this is conveniently written:

$$-\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}$$

$$- \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Here is another way to derive this formula: In Cartesian coordinates, we can derive $-\nabla^2$ by writing

$$I = \int d^3x \quad \vec{\nabla}\alpha \cdot \vec{\nabla}\beta$$

assuming that $\alpha, \beta \to 0$ at infinity, and integrating by parts

$$I = \int d^3x \, [-\alpha(\nabla^2\beta)]$$

we identify $(-\nabla^2\beta)$ as the coefficient of α under the integral.

Try this in spherical coordinates. The volume element is the product of the three line elements along mutually orthogonal directions

$$dv = d^3x = dr\,(r\,d\theta)\,(r\sin\theta\, d\phi) = dr\, d\theta\, d\phi\, r^2 \sin\theta$$

$$I = \int dr\, d\theta\, d\phi\; r^2\sin\theta \quad \vec{\nabla}\alpha \cdot \vec{\nabla}\beta$$

$$= \int dr\, d\theta\, d\phi\; r^2 \sin\theta \left\{ \frac{\partial\alpha}{\partial r}\frac{\partial\beta}{\partial r} + \frac{1}{r^2}\frac{\partial\alpha}{\partial\theta}\frac{\partial\beta}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial\alpha}{\partial\phi}\frac{\partial\beta}{\partial\phi} \right\}$$

Now integrate by parts, dropping surface terms.

$$= \int dr\, d\theta\, d\phi \left\{ \alpha \left(-\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial}{\partial r}(\beta)\right)\right) + \alpha\left(-\frac{\partial}{\partial\theta} \frac{r^2\sin\theta}{r^2} \frac{\partial}{\partial\theta}(\beta)\right) + \alpha\left(-\frac{\partial}{\partial\phi} \frac{r^2 \sin\theta}{r^2\sin^2\theta} \frac{\partial}{\partial\phi}\beta\right)\right\}$$

now form back the volume element by pulling $r^2\sin\theta$ to the left. But, note that r^2 cannot be pulled thru $\frac{\partial}{\partial r}$, and that $\sin\theta$ cannot be pulled through $\frac{\partial}{\partial\theta}$. Thus:

$$I = \int dr\, d\theta\, d\phi\, r^2\sin\theta \cdot \alpha \cdot \left\{ -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\beta$$

and we identify this operator as $-\nabla^2$, in agreement with the result on p. 5.

Now we can try to solve Laplace's equation

$$-\nabla^2\Phi = 0$$

or

$$\left(-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\varphi = 0$$

I will describe a solution in terms of a series of Sturm-Liouville problems. Start with ϕ. Consider the system:

$\mathcal{V}$ = function periodic under $\phi \to \phi + 2\pi$

$$\mathcal{O} = -\frac{\partial^2}{\partial\phi^2} \qquad \langle f, g\rangle = \int_0^{2\pi} d\phi\; f\cdot g$$

Θ is a self-adjoint operator:

$$\langle f, \Theta g\rangle = \int_0^{2\pi} d\phi \; f\left(-\frac{d^2}{d\phi^2} g\right)$$

$$= -f \frac{dg}{d\phi}\Big|_0^{2\pi} + \int_0^{2\pi} d\phi \; \frac{df}{d\phi}\frac{dg}{d\phi} \quad \cdots$$

$$= \left[-f\frac{dg}{d\phi} + \frac{df}{d\phi}\cdot g\right]\Big|_0^{2\pi} + \int_0^{2\pi} d\phi \left(-\frac{d^2 f}{d\phi^2}\right) g$$

If f and g are periodic, the surface term vanishes and

$$= \langle \Theta f, g\rangle$$

The eigenvectors of Θ are sin's and cos's. Often, these eigenvectors are written as the complex linear combinations:

$$f_m(\phi) = e^{im\phi} = \cos m\phi + i \sin m\phi$$

$$\Theta \, f_m(\phi) = m^2 f_m(\phi)$$

For periodicity, m must be an integer. The functions

$$e^{im\phi} \quad e^{-im\phi} \quad \text{or} \quad \cos m\phi \quad \sin m\phi$$

are the unique eigenvectors with eigenvalue $\underline{m^2}$.

By the Sturm-Liouville theorem, we can expand

$$\varphi = \sum_m \psi_m(r,\theta)\, e^{im\phi}$$

When we act $-\nabla^2$ on φ, the coefficients of $e^{im\phi}$ must vanish separately, since these are orthogonal functions. The action of $-\nabla^2$ on the coefficient $\psi_m(r,\theta)$ gives the equation:

$$\left(-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta}\right]\right)\psi_m = 0$$

The θ equation defines another Stürm-Liouville problem:

$\mathcal{V}$ = functions of θ on the interval $[0,\pi]$

$$\langle f,g\rangle = \underbrace{\int d\theta \sin\theta}_{\theta \text{ part of volume element}} \; f(\theta)\, g(\theta)$$

$$\Theta = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta}$$

For $m=0$, I will insist that f, g are simply square-integrable functions on the interval. For $m\neq 0$, I will require $f,g = 0$ at $\theta = 0,\pi$. This had better be true, in order for φ to be well-defined, since $e^{im\phi}$ is ill-defined at the north and south poles for $m\neq 0$.

To Check that Θ is self-adjoint, we need

$$\int_0^\pi d\theta \sin\theta \; f\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}g\right) = \int_0^\pi d\theta \,(-f)\left(-\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}g\right)$$

$$= f(\theta)\sin\theta \frac{dg}{d\theta}\Big|_0^\pi + \int_0^\pi d\theta \frac{df}{d\theta}\sin\theta\frac{dg}{d\theta}$$

$$= \Big[\underbrace{f(\theta)\sin\theta\frac{dg}{d\theta} - \frac{df}{d\theta}\sin\theta\, g(\theta)}_{=0 \text{ since } \sin\theta = 0 \text{ at } 0,\pi}\Big]\Big|_0^\pi + \int d\theta\left(-\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}f\right) g$$

$$= \int_0^\pi d\theta \sin\theta \left[-\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} f\right] g$$

so

$$\langle f, \mathcal{O}g\rangle = \langle \mathcal{O}f, g\rangle.$$

Eventually, you will need to understand this whole class of functions, but I would like to start with the very important subclass $m = 0$. The terms with $m=0$ correspond to cylindrically symmetric solutions to Laplace's equation. The find the orthogonal functions, we must solve the eigenvalue problem:

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} f = \lambda f(\theta)$$

It is very convenient to write $x = \cos\theta$ $\quad dx = \sin\theta\, d\theta$, so the above equation becomes

$$\mathcal{O}f = -\frac{d}{dx}(1-x^2)\frac{d}{dx} f(x) = \lambda f(x)$$

We can find solutions to this equation which are polynomials.

$$p_0(x) = 1 \quad \text{satisfies} \quad \Theta p = \lambda p \quad \text{with } \lambda = 0$$

$$p_1(x) = x \quad \text{satisfies} \quad \Theta p = \lambda p \quad \text{with } \lambda = 2$$

$$\Theta x^2 = -\frac{d}{dx}(1-x^2)\frac{d}{dx} x^2 = -\frac{d}{dx}\left((1-x^2)\cdot 2x\right) = 6x^2 - 2$$

so $\quad p_2(x) = (x^2 - \frac{1}{3})$ satisfies with $\quad \lambda = 6$

similarly $\quad p_3(x) = (x^3 - \frac{3}{5}x)$ satisfies with $\quad \lambda = 12$

In general, if we start with $p_\ell(x) = x^\ell$ + lower order

we can see that

$$\Theta x^\ell = \ell(\ell+1) x^\ell + \text{terms of lower order}$$

and it is possible to adjust the lower order terms to obtain an eigenfunction with eigenvalue $\lambda = \ell(\ell+1)$. These polynomials are conventionally defined with a prefactor that makes them equal to 1 at $x = 1$. Then they are the

<u>Legendre polynomials</u>

$P_\ell(x)$ s.t. $\quad P_\ell(x) \sim x^\ell$ + lower powers.

$$P_\ell(1) = 1$$

$$\left[-\frac{d}{dx}(1-x^2)\frac{d}{dx}\right] P_\ell(x) = \ell(\ell+1)\, P_\ell(x)$$

The $P_\ell(x)$ are the only eigenfunctions of Θ that are finite at $x = 1, -1$ $(\theta = 0, \pi)$. Thus it follows that:

① the $P_\ell(x)$ are orthogonal:

$$\langle P_\ell(x), P_{\ell'}(x) \rangle = \int_{-1}^{1} dx \; P_\ell(x) P_{\ell'}(x) = 0 \quad \text{for } \ell \neq \ell'$$

② the $P_\ell(x)$ are a complete set; for any function of $x \in [-1, 1]$

$$f(x) = \sum_{\ell=0}^{\infty} f_\ell \, P_\ell(x)$$

with

$$\int_{-1}^{1} dx \left(f(x) - \sum_{0}^{L} f_\ell P_\ell(x) \right)^2 \longrightarrow 0 \quad \text{as } L \to \infty$$

so we can represent a general ~~[illegible]~~ cylindrically symmetric function as

$$\varphi(r, \theta) = \sum_\ell \chi_\ell(r) \, P_\ell(\cos\theta)$$

acting $-\nabla^2$ on this φ and using the fact that the P_ℓ are eigenvectors of a part of $-\nabla^2$ and mutually orthogonal, we obtain the equations

$$\left[-\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} \right] \chi_\ell(r) = 0$$

It is easy to recognize that the functions

$$r^\ell \qquad \left(\frac{1}{r}\right)^{\ell+1}$$

solve this equation. Since a second-order differential equation has a solution specified uniquely by two parameters, we now have the most generally cylindrically symmetric solution of Laplace's equation:

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

The P_l are exceptionally useful functions, and I will say more about them in a moment. But first I would like to finish the story for non-cylindrically symmetric functions ($m \neq 0$). The eigenfunctions of the $m \neq 0$ operator

$$\left[-\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \frac{m^2}{\sin^2\theta} \right] f(\theta) = \lambda f(\theta)$$

with $f(\theta) = 0$ at $\theta = 0, \pi$, exist only for

$$\lambda = l(l+1) \qquad \text{with, further} \quad l \geq |m|$$

The standard form of f is called the <u>associated Legendre function</u> $P_l^m(x)$ $\quad x = \cos\theta$. The full dependence on θ, φ can also be expressed as the combination

$$Y_{lm}(\theta,\phi) = \eta_{lm} \, P_l^m(\cos\theta) \, e^{im\phi}$$

where η_{lm} is an overall factor to insure:

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \; |Y_{lm}(\theta,\phi)|^2 = 1$$

The Y_{lm} are called spherical harmonics. They are orthogonal in the measure above

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \; (Y_{lm}(\theta,\varphi))^* (Y_{\bar{l}\bar{m}}(\theta,\varphi)) = \delta_{l\bar{l}}\, \delta_{m\bar{m}}$$

For $m=0$

$$Y_{l0} = \left[\frac{2l+1}{4\pi}\right]^{\frac{1}{2}} P_l(\cos\theta)$$

[$Y_{l0} \propto P_l(\cos\theta)$ should be clear; the prefactor is not obvious, but I will prove it in a moment.]. Since the eigenvalue of Y_{lm} is $l(l+1)$ for any m, the solution of Laplace's equation generalizes to

$$\varphi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^{l} + B_{lm}\frac{1}{r^{l+1}}\right) Y_{lm}(\theta,\varphi)$$

since $l \geq |m|$

This is a representation of the most general solution of Laplace's equation.

The functional forms of the $Y_{\ell m}$ and P_ℓ are tabulated in quantum mechanics textbooks. I'll leave $Y_{\ell m}$ for next year, but I would like you to become familiar with the P_ℓ. Here is an elegant treatment of the properties of the P_ℓ:

Consider the function $$T(x,s) = \frac{1}{[1-2xs+s^2]^{\frac{1}{2}}}$$

I would like to propose an alternative definition of the P_ℓ as the coefficients in the Taylor series expansion of T:

$$T(x,s) = 1 + s\,x + s^2\left(\frac{3x^2-1}{2}\right) + \cdots$$

You can check that the coefficient of s^ℓ is a polynomial in x which is even if ℓ is even and odd if ℓ is odd. Then define the P_ℓ by

$$T(x,s) = \sum_\ell s^\ell P_\ell(x)$$

T is called the <u>generating function</u> of the P_ℓ.

Note that
$$T(x=1,s) = \frac{1}{[1-2s+s^2]^{\frac{1}{2}}} = \frac{1}{1-s} = 1+s+s^2+s^3+\cdots$$

so this definition implies $P_\ell(x=1) = 1$ as required

Next, you can prove by painful explicit differentiation that

$$\left(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\right)T(x,s)$$

$$= \frac{2xs - 3s^2 - x^2s^2 + 2xs^3}{[1-2xs+s^2]^{5/2}}$$

$$= s\frac{d^2}{ds^2}\, s\, T(x,s)$$

apply this identity to the series, we have

$$\sum_\ell s^\ell \left(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\right)P_\ell(x) = \sum_\ell \ell(\ell+1)\, s^\ell\, P_\ell(x)$$

so indeed

$$-\frac{d}{dx}(1-x^2)\frac{d}{dx}P_\ell(x) = \ell(\ell+1)\,P_\ell(x)$$

Finally,

$$\int_{-1}^{1} dx\; T^2(x,s) = \int_{-1}^{1} dx\; \frac{1}{1-2xs+s^2}$$

$$= \frac{1}{2s}\log\left(\frac{1+2s+s^2}{1-2s+s^2}\right)$$

$$= \frac{1}{s}\log\left(\frac{1+s}{1-s}\right) = \frac{1}{s}\left[\log(1+s) - \log(1-s)\right]$$

so that

$$\int_{-1}^{1} dx \; T^2(x,s) = 2 + \frac{2}{3}s^2 + \frac{2}{5}s^4 + \frac{2}{7}s^6 + \cdots$$

If we remember that $\int_{-1}^{1} dx \, P_\ell(x)\, P_{\ell'}(x) = 0$ for $\ell \neq \ell'$

the left-hand side is

$$\int_{-1}^{1} dx \sum_{\ell} s^{2\ell} P_\ell^2(x)$$

and we can identify

$$\int_{-1}^{1} dx \; P_\ell^2(x) = \frac{2}{2\ell+1}$$

The first few Legendre polynomials are:

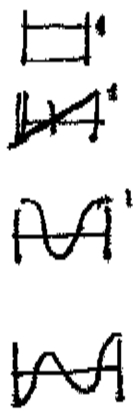
$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{3x^2-1}{2}$$

$$P_3(x) = \frac{5x^3 - 3x}{2}$$

$$\vdots$$



For completeness, I'll list the first few spherical harmonics also:

$$Y_{00} = \frac{1}{(4\pi)^{1/2}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\,\cos\theta \qquad Y_{1-1} = \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\phi}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{i2\phi} \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\;\frac{3\cos\theta - 1}{2}$$

$$Y_{2-1} = \sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{-i\phi} \qquad Y_{2-2} = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{-2i\phi}$$

$$\vdots$$

Now that we have a nice formula for the solution of Laplace's equation for a cylindrically symmetric problem, we can use it to solve some electrostatic problems. Consider first the problem at the beginning of this lecture: We have a sphere with $\Phi = V/2$ in the northern hemisphere and $\Phi = -V/2$ in the southern hemisphere. Let

$$\varepsilon(\cos\theta) = \begin{cases} +1 & \theta < \pi \quad \cos\theta > 0 \\ -1 & \theta > \pi \quad \cos\theta < 0 \end{cases}$$

We can write $\varepsilon(\cos\theta)$ in the basis of orthogonal functions given by the Legendre polynomials.

$$\varepsilon(x) = \sum_{l=0}^{\infty} e_l\, P_l(x)$$

actually, $\varepsilon(x)$ is odd, so only term w. ℓ odd can appear. The coefficients are given by

$$\langle P_\ell(x), \varepsilon(x) \rangle = \sum_{\substack{\bar{\ell}=1 \\ \text{odd}}}^{\infty} e_{\bar{\ell}} \langle P_\ell(x), P_{\bar{\ell}}(x) \rangle$$

the inner product is

$$\langle f, g \rangle = \int_{-1}^{1} dx \; f(x)\, g(x) \quad \left(\text{recall } \int d\theta \sin\theta = \int d(\cos\theta) = \int dx \right)$$

so

$$\int_{-1}^{1} dx \; \underbrace{P_\ell(x)\, \varepsilon(x)}_{} = e_\ell \frac{2}{2\ell+1}$$

$= 0$ for ℓ even,

for ℓ odd:

$$2 \int_0^1 dx \; P_\ell(x) = \frac{2}{2\ell+1} e_\ell$$

Putting in the first two odd Legendre polynomials from p. 16

$P_1 = x \qquad 2 \cdot \frac{1}{2} = \frac{2}{3} e_1 \quad \Rightarrow \quad e_1 = \frac{3}{2}$

$P_3 = \frac{5x^3 - 3x}{2} \qquad 2 \cdot \left(-\frac{1}{8}\right) = \frac{2}{7} e_3 \quad \Rightarrow \quad e_3 = -\frac{7}{8}$

$\vdots$

[Higher terms can be obtained from

$$\int_0^1 dx\; T(x,s) = \int_0^1 dx \frac{1}{[1-2xs+s^2]^{1/2}} = -\frac{1}{s}\left(1-2xs+s^2\right)^{1/2}\Big|_0^1$$

$$= 1 + \frac{1}{s}\left(\sqrt{1+s^2} - 1\right)$$

$$= 1 + \frac{1}{2}s - \frac{1}{8}s^3 + \frac{1}{16}s^5 - \frac{5}{128}s^7 + \cdots$$

so that

$$\int_0^1 dx\; P_\ell(x) = \begin{array}{ccccc} \frac{1}{2} & -\frac{1}{8} & \frac{1}{16} & -\frac{5}{128} & \cdots \\ \ell = 1, & 3, & 7, & 9, & \cdots \end{array}$$]

In any event, our boundary condition is

$$\phi(r=R, \theta) = \frac{V}{2}\cdot\left(\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \cdots\right)$$

$$x = \cos\theta$$

Compare this to the general solution

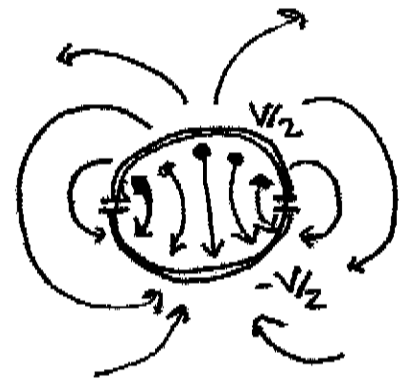
$$\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

The P_ℓ are orthogonal, so we can just match terms.
For the region outside the sphere, $\phi(r,\theta)$ should $\to 0$ as $r \to \infty$. Thus, all $A_\ell = 0$ and we have $\underline{r > R}$

$$\phi(r,\theta) = V\cdot\left[\frac{3}{4}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{7}{16}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) + \cdots\right]$$

For the region inside the sphere, $\Phi(r,\theta)$ must have no singularity at $r=0$. Thus, all of the B_ℓ must be zero, and so we find $r<R$

$$\Phi(r,\theta) = V\cdot\left[\frac{3}{4}\left(\frac{r}{R}\right)P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \cdots\right]$$



A closely related problem is that of a conducting sphere immersed in a background electric field. Say that far from the sphere

$$\vec{E} = -\hat{z}\,\vec{E}_0 \qquad \Phi = zE_0 = rE_0\cos\theta$$

then

$$\Phi = (\text{const}) \text{ at } r=R$$

$$\Phi = rE_0 P_1(\cos\theta) + (\text{terms} \to 0 \text{ as } r\to\infty)$$

compare to the general solution:

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta)$$

$$= A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2}\right) P_1(\cos\theta)$$

$$+ \left(A_2 r^2 + \frac{B_2}{r^3}\right) P_2(\cos\theta) + \cdots$$

A_0 is an arbitrary constant; we can set it to 0.

$4\pi\epsilon_0 B_0$ is the charge on the sphere. If the sphere is neutral, $B_0 = 0$

$A_2, A_3, A_4 \cdots$ must $= 0$ if $\phi \sim r^1 E_0$ as $r \to \infty$

and $A_1 = E_0$

Finally, if we match that $\phi = \text{const}$ on the surface $r = R$, all of the $B_l = 0$ except for B_1, which satisfies

$$\left(E_0 r + \frac{B_1}{r^2}\right)\Big|_{r=R} = 0$$

thus.

$$\varphi(r,\theta) = \left(E_0 r - E_0 \frac{R^3}{r^2}\right) \cos\theta$$

This is the same solution that we derived earlier by the method of images:

$$\varphi = E_0 \left(z - \frac{R^3 z}{r^3} \right)$$