Solutions of Laplace's Equation - 2

Oct. 16

Before leaving this example of the use of Fourier series to solve Laplace's equation, I'd like to point out some pathologies.

First, the series that I gave for C does not actually converge, since

$$\frac{\cosh \lambda_{nm} d/2}{\sinh \lambda_{nm} d/2} \to 1 \qquad (\lambda_{nm} \to \infty)$$

for $n, m \to \infty$. However, differentiating with respect to d gives

$$\frac{d}{dd}\left(\frac{\cosh \lambda_{nm} d/2}{\sinh \lambda_{nm} d/2}\right) = -\frac{\lambda_{nm}}{2}\,\frac{1}{\sinh^2(\lambda_{nm} d/2)}$$

$$\frac{dC}{dd} = \sum_{\substack{n,m \\ \text{odd}}} -\frac{16\epsilon_0}{\pi^2}\,\frac{n^2+m^2}{n^2 m^2}\left(\frac{1}{\sinh^2 \lambda_{nm} d/2}\right) < 0$$

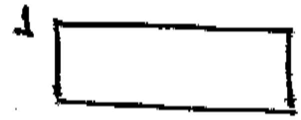
and $\to 0$ exponentially for $d \gg a$

The d-independent capacitance is infinite because we have an idealized join:

which can store an infinite amount of charge. A realistic smooth corner would give finite C for $d \to \infty$ but makes a harder problem to solve.

Second, the Fourier representation of ϕ must also be used with care. Going back to 1-dimension, we represented the function

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & 0 < x < a \\ 0 & x = a \end{cases}$$



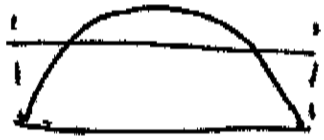
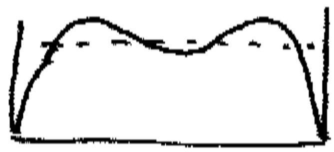
as the Fourier series

$$f(x) = \sum_{n\ \mathrm{odd}} \frac{4}{\pi n} \sin \frac{n\pi x}{a}$$

[check: $\langle \sin \frac{\pi m x}{a}, f(x) \rangle = \frac{2a}{\pi m}$ m odd

$$= \sum_{n\ \mathrm{odd}} \frac{4}{\pi n} \underbrace{\langle \sin \frac{\pi m x}{a}, \sin \frac{\pi n x}{a} \rangle}_{\frac{a}{2}\delta_{mn}} \quad \checkmark$$]

The representation of $f(x)$ is basically sensible:

1 term:  , 2 terms:  , and

etc.

But, it doesn't work everywhere. Consider the series with (N+1) terms

$$f_N(x) = 4\left\{ \frac{1}{\pi}\sin\frac{\pi x}{a} + \frac{1}{3\pi}\sin\frac{3\pi x}{a} + \cdots + \frac{1}{(2N+1)\pi}\sin\frac{(2N+1)\pi x}{a} \right\}$$

Evaluate this at the maximum of the middle sin function

$$x = \frac{a}{2N+1}$$

$$f_N\left(\frac{a}{2N+1}\right) = 4\left\{ \frac{1}{\pi}\sin\frac{\pi}{2N+1} + \frac{1}{3\pi}\sin\frac{3\pi}{2N+1} + \cdots + \frac{1}{(2N+1)\pi}\sin\frac{(2N+1)\pi}{2N+1} \right\}$$

$$= \frac{4}{(2N+1)}\left\{ \frac{2N+1}{\pi}\sin\frac{\pi}{2N+1} + \frac{2N+1}{3\pi}\sin\frac{3\pi}{2N+1} + \cdots + \frac{1}{\pi}\sin\pi \right\}$$

There are (N+1) terms, all of order $\frac{1}{2N+1}$, adding to an answer of order 1.

So let

$$t = \frac{(2m+1)}{2N+1}\pi$$

and replace the sum over m by an integral over t

$$dt = \frac{2\pi}{2N+1}\,dm \qquad \text{or} \qquad \sum_m 1 = \sum_m dm = \int_0^\pi dt\,\frac{2N+1}{2\pi}$$

then

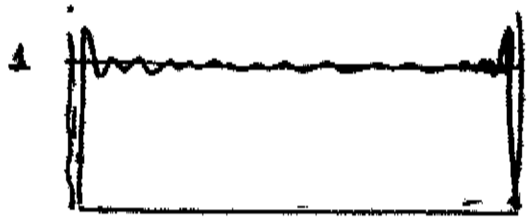
$$f_N\left(\frac{a}{2N+1}\right) \xrightarrow[N\to\infty]{} \frac{4}{2N+1}\,\frac{2N+1}{2\pi}\int_0^\pi dt\,\frac{1}{t}\sin t$$

$$= \frac{4}{2\pi}\int_0^\pi dt\,\frac{1}{t}\sin t$$

$$= \frac{2}{\pi}\,Si(\pi)$$ Sine integral Abramowitz + Stegun chapter 5

$$= \frac{2}{\pi}(1.85194) = 1.179$$

so $f_N(x)$ always overshoots the value 1 at some point even as $N \to \infty$. This behavior is called "Gibbs phenomenon". The typical behavior of $f_N(x)$ for large N is:

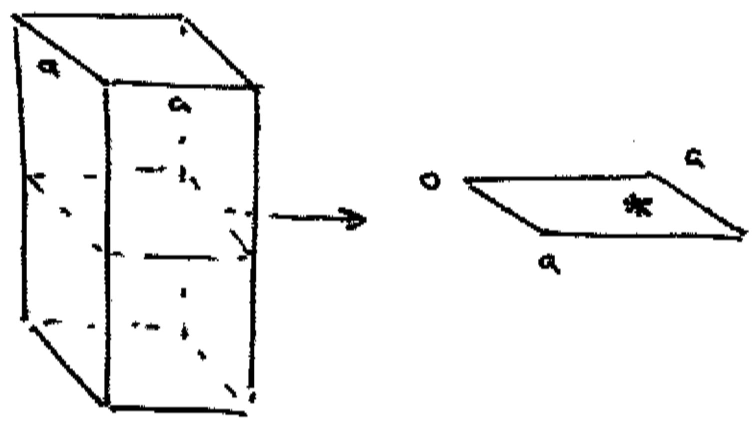


(see Griffiths, Fig. 3.19)

overshoot in a region of size $\frac{1}{N}$; good approx. elsewhere

Let's solve one more electrostatics problem with Fourier series, and then I will talk more about the general structure of the method:

Consider a pipe of square cross section with grounded walls ($\phi = 0$)



and put a charge Q at (x_0, y_0) on the plane $z = 0$.

The problem is to find ϕ and $\vec{E}$ in the pipe.

We can solve this problem using an infinite number of image charges, but we can also solve it by Fourier series. Since $\phi = 0$ at $x = 0, a$ $y = 0, a$, we can represent ϕ as:

$$\phi = \sum_{n,m=1}^{\infty} \alpha_{nm}(z) \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}$$

The equation $-\nabla^2\phi = 0$ becomes:

$$\left(\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{a}\right)^2 + \left(-\frac{d^2}{dz^2}\right) \right) \alpha_{nm}(z) = 0$$

for each Fourier component. Defining $\lambda_{nm} = \frac{\pi}{a}[n^2+m^2]^{\frac{1}{2}}$ as before

$$\frac{d^2}{dz^2} \alpha_{nm}(z) = \lambda_{nm}^2 \alpha_{nm}(z)$$

The solution is $(z > 0)$

$$\alpha_{nm}(z) = A_{nm} e^{-\lambda_{nm} z} + B e^{+\lambda_{nm} z}$$

since ϕ should $\rightarrow 0$ as $z \rightarrow +\infty$, $B = 0$

The problem is symmetrical for negative z so.

$$\phi = \sum_{n,m=1}^{\infty} A_{nm} e^{-\lambda_{nm}|z|} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}$$

and now we only need to find the A_{nm}.

Since the charge is specified, this is a problem with Neumann boundary conditions at $z=0$. The problem is symmetrical about $z=0$, so

and
$$\rho = Q\,\delta(x-x_0)\,\delta(y-y_0)$$

This δ-function boundary condition has a simple Fourier representation:

$$\rho = \sum_{n,m=1}^{\infty} r_{nm} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}$$

where we can find the coefficients by

$$\left\langle \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}, \rho \right\rangle = \sum_{\bar{n}\bar{m}=1}^{\infty} r_{\bar{n}\bar{m}} \left\langle \sin\frac{\pi \bar{n} x}{a} \sin\frac{\pi \bar{m} y}{a}, \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a} \right\rangle$$

$$= \sum_{\bar{n}\bar{m}} r_{\bar{n}\bar{m}}\ \frac{a}{2}\delta_{n\bar{n}}\ \frac{a}{2}\delta_{m\bar{m}}$$

so
$$\left(\frac{a}{2}\right)^2 r_{nm} = \left\langle \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}, \rho \right\rangle$$

This is ~~easily~~ evaluated:

$$= \int dx\,dy\ \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}\ Q\,\delta(x-x_0)\,\delta(y-y_0)$$

So

$$\left(\frac{a}{2}\right)^2 r_{nm} = Q \sin\frac{\pi n x_0}{a} \sin\frac{\pi m y_0}{a}$$

so

$$\rho(x,y) = \sum_{nm=1}^{\infty} Q\left(\frac{2}{a}\sin\frac{\pi n x_0}{a}\right)\left(\frac{2}{a}\sin\frac{\pi m y_0}{a}\right) \cdot \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}$$

[and $\delta(x-x_0) = \sum_{m=1}^{\infty} \frac{2}{a}\sin\frac{\pi n x_0}{a} \sin\frac{\pi n x}{a}$,

a rather simple result.]

this should equal

$$2\epsilon_0 E^z(z=0^+) = -2\epsilon_0 \frac{\partial}{\partial z}\phi(x,y,z)\Big|_{0^+}$$

$$= \sum_{nm} 2\epsilon_0 \lambda_{nm} A_{nm} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}$$

so

$$A_{nm} = \frac{4Q}{a^2}\frac{1}{2\epsilon_0\lambda_{nm}} \sin\frac{\pi n x_0}{a} \sin\frac{\pi m y_0}{a}$$

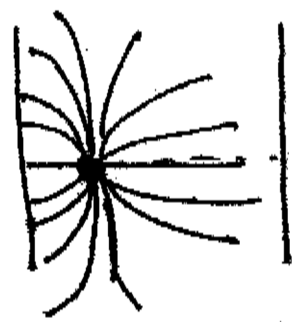
$$= \frac{2Q}{\pi a\epsilon_0}\frac{1}{[n^2+m^2]^{1/2}} \sin\frac{\pi n x_0}{a} \sin\frac{\pi m y_0}{a}$$

$\doteq$ all!

$$\phi = \sum_{n,m=1}^{\infty} \frac{2Q}{\pi \epsilon_0 a} \frac{1}{[n^2+m^2]^{1/2}} \sin\frac{\pi n x_0}{a} \sin\frac{\pi m y_0}{a}$$

$$\cdot\, e^{-\gamma_{nm}|z|} \sin\frac{\pi n x}{a} \sin\frac{\pi y}{a}$$

We could have anticipated that the potential ϕ would be smaller if the source were placed closer to the walls, and that it would decrease as $|z| \to \infty$



The surprise is ~~that~~ ϕ and $\vec{E}$ fall off exponentially and that at large z the potential has the form

$$\phi \sim \left[\frac{2Q}{\pi \epsilon_0 a} \sin\frac{\pi x_0}{a} \sin\frac{\pi y_0}{a}\right] \cdot \left(e^{-\frac{\pi}{a} z} \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}\right)$$

which is independent of the initial placement of the charge except in the overall magnitude of the coefficient.

We have now seen in practical terms how to solve electrostatics problems using Fourier series. Now let's look further at the foundations. Why does this trick work, and how do we generalize it?

The basis of the trick is a remarkable, general result called the Sturm-Liouville theorem:

Let $\mathcal{V}$ be a vector space with elements f, which might be finite dimensional vectors or functions — viewed here as infinite-dimensional vectors.

Let $\langle f, g \rangle$ be an inner product on this vector space: $\langle f, g\rangle$ is real, and $\langle f, f\rangle > 0$ if $f \neq 0$.
$\langle f, g\rangle = \langle g, f\rangle$ $\quad \langle f, \alpha g_1 + \beta g_2\rangle = \alpha\langle f, g_1\rangle + \beta\langle f, g_2\rangle$.

Let $\mathcal{O}$ be a linear operator on vectors:

if $f \in \mathcal{V}$ $\quad \mathcal{O}f \in \mathcal{V}$ $\qquad \mathcal{O}(\alpha f + \beta g) = \alpha\,\mathcal{O}f + \beta\,\mathcal{O}g$.

and let $\mathcal{O}$ be self-adjoint with respect to the inner product:

$$\langle f, \mathcal{O}g\rangle = \langle \mathcal{O}f, g\rangle$$

Finally, define an eigenvector of $\mathcal{O}$ as a vector f_i

st. $\qquad \mathcal{O}f_i = \lambda_i f_i \qquad \lambda_i \in \mathbb{R}$

λ_i is called the eigenvalue

Then the collection of f_i form a basis for $\mathcal{V}$, and any element $g \in V$ can be written as a series in the f_i:

$$g = \sum_i \alpha_i f_i$$

in the sense ~~that~~

$$\lim_{N\to\infty} \langle g - \sum_{i=1}^{N} \alpha_i f_i \, , \; g - \sum_{1}^{N} \alpha_i f_i \rangle \to 0$$

as the no. of terms $N \to \infty$.

This theorem is strong and forbidding. But let me give a ~~full~~ example with finite-dimensional vectors and matrices. Consider a space of 3-dimensional vectors, with

$$\langle v, w \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Θ is a 3x3 matrix: $\langle v, \Theta w \rangle = v^T \Theta w = v_i (\Theta_{ij} w_j)$

We can rewrite this as

$$\langle v, \Theta w \rangle = v_i \Theta_{ij} w_j = \Theta_{ij} v_i \, w_j = ((\Theta^T)_{ji} v_i) w_j$$

$$= \langle \Theta^T v, w \rangle$$

so Θ is self-adjoint if it is a symmetric matrix

Here is an example:

$$\Theta = \begin{pmatrix} 6 & -2 & 4 \\ -2 & 9 & -8 \\ 4 & -8 & 26 \end{pmatrix}$$

You can check that

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{is an eigenvector with} \quad \lambda_1 = 5$$

$$v_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \quad \text{is an eigenvector with} \quad \lambda_2 = 6$$

$$v_3 = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{is an eigenvector with} \quad \lambda_3 = 30$$

Note that v_1 v_2 v_3 are mutually orthogonal. This follows from our general assumptions:

Let v_1 v_2 be eigenvectors of <u>self-adjoint</u> O with <u>distinct</u> eigenvalues λ_1 λ_2 then

$$\langle v_1, O v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

Notice that self-adjointness is essential!

$$= \langle O v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle$$

so

$$(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0 \Rightarrow \lambda_1 = \lambda_2 \text{ or } \langle v_1, v_2 \rangle = 0$$

The Sturm-Liouville theorem says that, given any $N \times N$ real symmetric matrix, we can find N orthogonal eigenvectors v_i. We can write any vector as a linear combination of these; hence any v can be represented as

$$v = \sum_1^N \alpha_i v_i$$

These ideas work in the same way for functions on an interval $[0,a]$ which vanish at the endpoints:

$\mathcal{V}$ = space of functions $f(x)$ s.t.

$f(x) = 0$ at $x = 0, a$ $\qquad \int_0^a dx\, f^2(x) < \infty$

We define

$$\langle f, g \rangle = \int_0^a dx\, f(x)\, g(x)$$

Now let

$$\mathcal{O} = -\frac{d^2}{dx^2}$$

$\mathcal{O}$ is self-adjoint with respect to $\langle\, , \rangle$:

$$\langle f, \mathcal{O} g \rangle = \int_0^a dx\, f(x) \left(-\frac{d^2}{dx^2}\right) g(x)$$

$$= \underbrace{f(x)\left(-\frac{d}{dx} g(x)\right)\Big|_0^a}_{=0 \text{ since } f=0 \text{ at the endpoints}} + \int_0^a dx\, \frac{df}{dx}\, \frac{dg}{dx}$$

$$= \underbrace{\frac{df}{dx}\, g(x)\Big|_0^a}_{=0 \text{ since } g=0 \text{ at the endpoints}} + \int_0^a dx \left(-\frac{d^2}{dx^2} f\right) \cdot g$$

$$= \langle \mathcal{O} f, g \rangle$$

The eigenvectors of $\mathcal{O}$ are the functions that satisfy

$$-\frac{d^2}{dx^2} f = \lambda_i f_i$$

for some λ_i, subject to $f(x) = 0$ at $x = 0, a$. The solutions to this problem are

$$f_n(x) = \sin(\sqrt{\lambda_n}\, x)$$

with $\sqrt{\lambda_n}\, a = n\pi$. That is,

$$f_n(x) = \sin\left(\frac{\pi n}{a} x\right) \qquad \lambda_n = \left(\frac{\pi n}{a}\right)^2$$

The general Sturm-Liouville theory then implies that we can expand any function $f(x)$ which vanishes at $x = 0, a$ as

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{\pi n x}{a}$$

i.e. in a Fourier series!

The Sturm-Liouville theory suggests a very general technique for solving Laplace's equation. With attention to the geometry of the problem and the boundary conditions, we write the Laplacian $(-\nabla^2)$ in terms of a self-adjoint operator. We then expand $\phi(\vec{x})$ in terms of the eigenvectors of this operator. As in the examples we have already discussed, Laplace's equation often simplifies dramatically in this representation.

In the next lecture, we'll use the Sturm-Liouville method to analyze Laplace's equation in spherical coordinates.