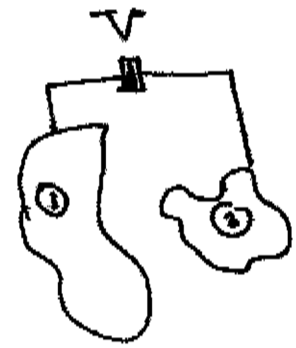
Solutions of Laplace's Equation

In the previous lecture, we discussed solutions to Laplace's and Poisson's equations that could be found by trickery. In the next several lectures, I will describe systematic methods for solving Laplace's equation. First, though, I would like to pause to introduce a last bit of electrostatic terminology.

Consider a situation with two conductors and a potential difference between them:



Charge will flow from 1 to 2 or from 2 to 1 until all of the electric fields in the conductor have been cancelled and we arrive at a solution of Laplace's equation with the boundary condition

$$\begin{aligned} \phi &= \phi_1 \text{ on the surface of } 1 \\ \phi &= \phi_2 \text{ on the surface of } 2 \end{aligned} \qquad \text{with } \phi_1 - \phi_2 = V$$

Let Q be the amount of charge that flows:

$$Q = \underbrace{Q_1}_{\text{total charge on } 1} = - \underbrace{Q_2}_{\text{total charge on } 2}$$

Then we define the capacitance of this set of conductors as

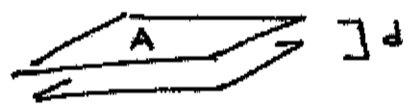
$$C = \frac{Q}{V}$$

by the linearity of electrostatics V is directly proportional to Q, or vice versa. An arrangement of materials ~~that~~ stores charge at a potential difference is called a capacitor. Capacitance is measured in Farads:

$$\text{Farad} = C/\text{Volt}.$$

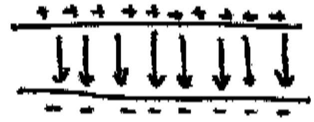
[Typically capacitors that are actually used in electric circuits are $1\mu F$ or smaller.]

The canonical capacitor consists of two parallel conducting plates. Consider an arrangement of area A and separation d, s.t. $A \gg d^2$



If there is a charge Q on the top plate and $-Q$ on the bottom, and the plate is so large that we can ignore the edges, the surfaces carry charge density

$$\rho = \pm \frac{Q}{A} \quad C/m^2$$



The electric field inside is constant, with magnitude

$$E = \frac{\rho}{\epsilon_0} = \frac{Q}{A\epsilon_0} \quad \downarrow$$

The potential difference is

$$V = \Delta\phi = -\int d\vec{x}\cdot\vec{E} = \frac{Q}{A\epsilon_0} d$$

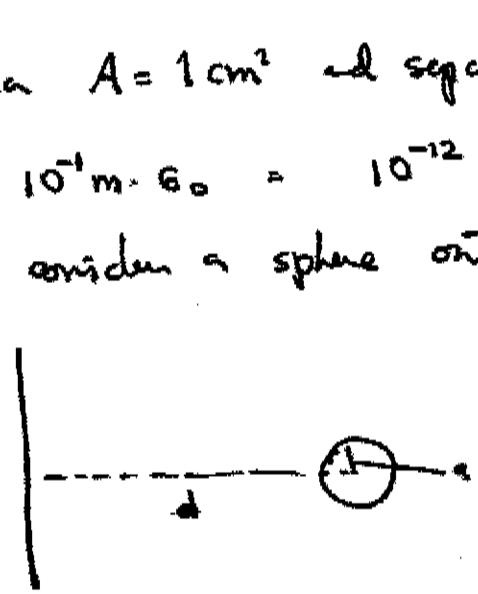
so the capacitance is

$$C = \frac{A\epsilon_0}{d} \qquad \text{Farad} = (\epsilon_0 \text{ units}) \cdot m$$

For two plates of area $A = 1\,cm^2$ and separation $d = 1\,mm$

$$C = 10^{-1} m \cdot \epsilon_0 = 10^{-12}\,F$$

As another example, consider a sphere outside of a conducting surface:



If $a \ll d$, we can consider a charge Q on the sphere to be a point charge. The solution of Laplace's equation gives an image charge at $z = -d$ and, physically, a charge $-Q$ on the surface. The electrostatic potential is

$$\phi = \frac{Q}{4\pi\epsilon_0 a} \text{ on the sphere} \qquad \phi = 0 \text{ on the surface}$$

so

$$V = \Delta\phi = \frac{Q}{4\pi\epsilon_0 a}, \qquad C = 4\pi a \epsilon_0$$

The capacitance of an electrostatic system appears in two interesting contexts, one involving energy, one involving time. Consider first the energy that must be expended to charge up the system, or equivalently, the energy stored in the capacitor. When the system has a separated charge q, the work needed to separate an additional charge dq across the potential difference $\Delta\phi$

$$dW = dq\,\Delta\phi = dq \cdot \frac{q}{C}$$

integrating this expression, the total work to charge the capacitor up to Q is

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2$$

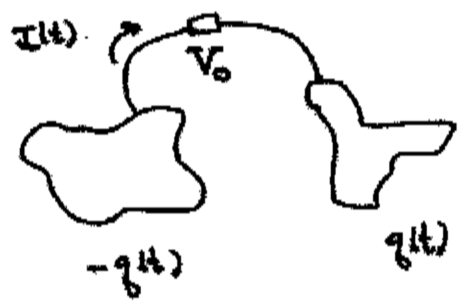
If $\phi(\vec{x})$, $\vec{E}(\vec{x})$ are the potential and field inside the capacitor when it is charged to $\Delta\phi = V$

$$C = \frac{2}{V^2} W(Q) = \frac{1}{V^2}\int d^2x\, \rho(x)\, \phi(x)$$

where ρ is the surface density of charge. . or

$$C = \frac{\varepsilon_0}{V^2}\int d^2x\, \hat{n}\cdot\vec{E}\,\phi = \frac{\varepsilon_0}{V^2}\int d^3x\, |\vec{E}|^2$$

Now think about the process by which stored charge is established. In this process, we have a situation



$$\frac{dq}{dt} = I$$

The potential difference driving the current is $V = V_0 - \frac{q}{C}$. If the conductors really have high conductivity, so that the dominant resistance is the resistance R of the wire, Ohm's law in the wire reads:

$$V_0 - \frac{q}{C_1} = RI = R\frac{dq}{dt}$$

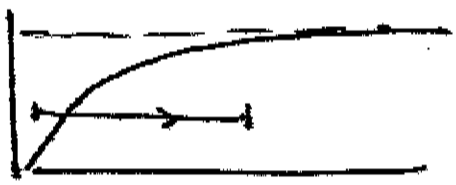
or

$$\frac{dq}{dt} = \frac{V_0}{R} - \frac{q}{RC}$$

If we start from $q(t)=0$ at $t=0$, the solution of the equation is

$$q(t) = V_0 C\left(1 - e^{-t/RC}\right)$$

that is:



characteristic time $\tau = RC_1$

units of $RC_1 = \frac{J/C}{C/sec}\cdot\frac{C}{J/C} = sec.$ ✓

The larger R or the larger C is, the longer it takes for the charge to fill in. In the example above of a 10^{-12} F capacitor, connecting across a $1000\,\Omega$ resistor gives a time constant

$$\tau \sim 1\ \text{nsec}.$$

When we computed the fields and potentials of a parallel-plate capacitor, we assumed that the area of the capacitor was large and ignored edge effects. What if we have a finite-size capacitor? How do we calculate the fields in that case?

Here is a tractable example: Consider a rectangular box:

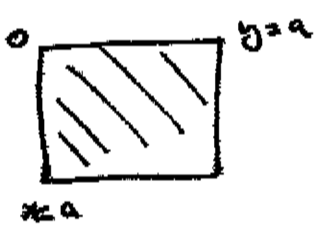
$\phi = +V/2$ on top

$\phi = -V/2$ on bottom

and $\phi = 0$ around the sides.

Let's calculate $\phi(\underline{x})$ in the interior. I will lay out a strategy for the calculation and carry it out; then I will put this strategy into a larger mathematical context.

Consider $\phi(x,y,z)$ as a function of x at fixed y and z.



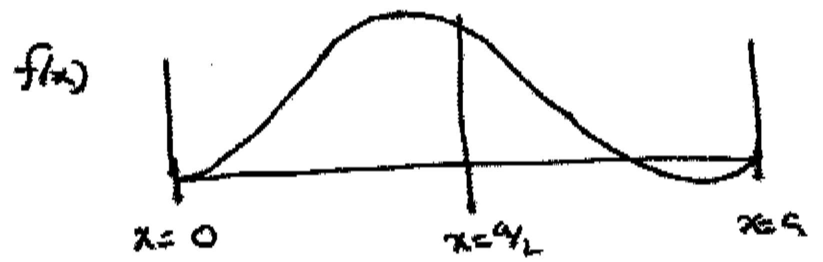
$$\phi(x,y,z) = 0 \quad \text{for } x=0 \,,\ x=a$$

Fourier's theorem states that any such function can be expanded in a series of sin and cos functions satisfying these boundary conditions:

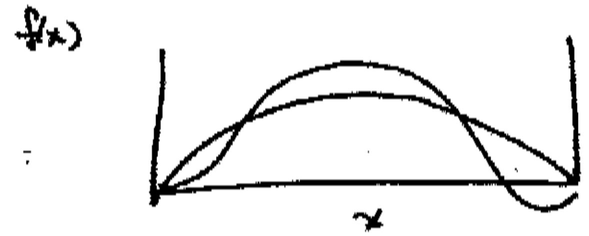
$$\underset{\text{at fixed } y,z}{\phi(x)} = \sum_{n=1}^{\infty} \alpha_n \underbrace{\sin\left(\frac{\pi n}{a}x\right)}$$

sin's vanishing at $x=0$, $x=a$

Intuitively, given a function of x which vanishes at $x=0$, $x=a$



we can approximate it in the mean by $A \sin \frac{\pi}{a} x$



approximate the difference of these functions by higher sin waves

$\sin \frac{2\pi}{a} x \quad \sin \frac{3\pi}{a} x$ terms.

until we get a very good approximation. There are two properties of this Fourier series which are important to note. First, we can define the dot product ("inner product") of two functions by:

$$\langle f, g \rangle = \int_0^a dx \; f(x)\, g(x)$$

[think of $f(x)$ as a vector whose components are the values of $f(x)$]

Then the sin functions appearing in the Fourier series are orthogonal

$$\langle \sin \tfrac{\pi n}{a} x , \sin \tfrac{\pi m}{a} x \rangle$$

$$= \int_0^a dx \; \sin \frac{\pi n}{a} x \; \sin \frac{\pi m}{a} x$$

$$= \int_0^a dx \; \frac{1}{2} \left[\cos \frac{\pi (n-m)}{a} x - \cos \frac{\pi (n+m)}{a} x \right]$$

where, if $n \neq m$, both cosines are integrated over a multiple of half a period, so the integrals vanish. For $n = m$, we get

$$\int_0^a dx \ \tfrac{1}{2} \cdot 1 = \frac{a}{2}$$

so we could even declare the functions

$$\sqrt{\frac{2}{a}} \ \sin \frac{\pi n}{a} x \qquad n = 1, 2, 3 \cdots$$

to be orthonormal vectors

$$\left\langle \sqrt{\frac{2}{a}} \sin \frac{\pi n}{a} x \ , \ \sqrt{\frac{2}{a}} \sin \frac{\pi m}{a} x \right\rangle$$

$$= \int_0^a dx \ \left(\sqrt{\frac{2}{a}} \sin \frac{\pi n}{a} x \right) \left(\sqrt{\frac{2}{a}} \sin \frac{\pi m}{a} x \right)$$

$$= \delta_{m,n}$$

Second, the Fourier series representation is accurate only in the sense that

$$\int_0^a dx \ \left(f(x) - \sum_{n=1}^{N} \alpha_n \sin \frac{\pi n}{a} x \right)^2 \longrightarrow 0$$

as we increase the number of terms N.

The Fourier series may overshoot or undershoot at specific points. I'll show you an example a bit later. Note that this condition is just

$$\left| f(x) - \sum_{1}^{N} a_n \sin \frac{\pi n}{a} x \right|^2 \to 0$$

where $\quad |g(x)|^2 = \langle g, g \rangle$

using the inner product we defined before.

If we can represent $\phi(x)$ as a Fourier series, we can do the same in y. Then we can write

$$\phi(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm}(z) \sin\left(\frac{\pi n}{a} x\right) \sin\left(\frac{\pi m}{a} y\right)$$

If we put this expression into Laplace's equation, we can see what is necessary to find a solution. Laplace's equation is linear, so operate on ϕ term by term:

$$-\nabla^2 \left[\alpha_{nm}(z) \sin \frac{\pi n}{a} x \ \sin \frac{\pi m}{a} y \right]$$

$$= \alpha_{nm}(z) \left[-\frac{\partial^2}{\partial x^2} \left(\sin \frac{\pi n}{a} x \right) \right] \sin \frac{\pi m}{a} y$$

$$+ \alpha_{nm}(z) \left(\sin \frac{\pi n}{a} x \right) \left[-\frac{\partial^2}{\partial y^2} \sin \frac{\pi m}{a} y \right]$$

$$+ \left(-\frac{\partial^2}{\partial z^2} \alpha_{nm}(z) \right) \sin \frac{\pi n}{a} x \ \sin \frac{\pi m}{a} y$$

but, note that $\quad \left(-\frac{\partial^2}{\partial x^2}\right) \sin \frac{\pi n}{a} x = \left(\frac{\pi n}{a}\right)^2 \sin\left(\frac{\pi n}{a} x\right)$

and similarly with y. So

$$-\nabla^2 \left[\alpha_{nm}(z) \sin\frac{\pi n x}{a} \sin\frac{\pi m}{a} y \right]$$

$$= \left[-\frac{\partial^2}{\partial z^2} \alpha_{nm} + \frac{\pi^2}{a^2}(n^2+m^2)\, \alpha_{nm} \right] \sin\frac{\pi n}{a} x \sin\frac{\pi m}{a} y$$

and this term, just by itself, solves Laplace's equation if

$$\frac{\partial^2}{\partial z^2} \alpha_{nm} = \frac{\pi^2}{a^2}(n^2+m^2)\, \alpha_{nm}$$

$$\alpha_{nm} = A_{nm} e^{\lambda_{nm} z} + B_{nm} e^{-\lambda_{nm} z}$$

$$\text{where} \qquad \lambda_{nm} = \frac{\pi}{a}\left[n^2+m^2\right]^{\frac{1}{2}}$$

you might think there could be other solutions in which there are cancellations between different terms in the Fourier series, but actually this is impossible. The reason is the orthogonality of the sin functions

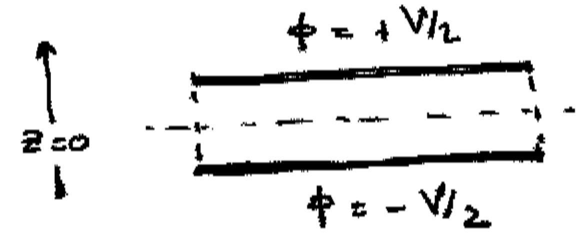
$$-\nabla^2\phi = 0 \quad \Rightarrow \quad \left\langle \sin\frac{\pi n}{a}x \,,\, -\nabla^2\phi \right\rangle = 0$$

$$\Rightarrow \text{coefficient of } \left(\sin\frac{\pi n}{a}x\right) \text{ in } \left(-\nabla^2\phi\right) = 0$$

so, we have found a representation of the most general solution of Laplace's equation with $\phi=0$ at $x=0, a$, $\phi = 0$ at $y = 0, a$:

$$\phi = \sum_{nm} \left(A_{nm} e^{\lambda_{nm} z} + B_{nm} e^{-\lambda_{nm} z} \right) \sin\frac{\pi n}{a} x \;\; \sin\frac{\pi m}{a} y$$

Now, what are the coefficients A_{nm}, B_{nm} for our example



Choose the origin of z in the center of the box. then $\phi(x,y,z)$ should be antisymmetric under inversion of z: $z \to -z$. This means $A_{nm} = -B_{nm}$ or

$$\phi = \sum_{n,m=1}^{\infty} 2A_{nm} \sinh\left(\lambda_{nm} z\right) \;\; \sin\frac{\pi n}{a} x \;\; \sin\frac{\pi m}{a} y$$

$$\lambda_{nm} = \frac{\pi}{a} \left[n^2 + m^2\right]^{\frac{1}{2}}$$

Finally, we need to determine the A_{nm} by matching this expression to the Dirichlet boundary condition

$$\phi(x,y, z = a/2) = \frac{V}{2} \qquad (\text{independent of } x,y)$$

we can do this match using orthogonality:

$$\left\langle \sin\frac{\pi \bar{n}}{a} x \; \sin\frac{\pi \bar{m}}{a} y \;,\; \phi(x,y,z) \right\rangle$$

$$= \int_0^a dx\, dy \;\; \sin\frac{\pi \bar{n}}{a} x \;\; \sin\frac{\pi \bar{m}}{a} y \;\; \sum_{nm} 2A_{nm} \sinh(\lambda_{nm} z) \sin\frac{\pi n}{a} x \sin\frac{\pi m}{a} y$$

$$= \sum_{nm} \left(\frac{a}{2}\right)^2 \delta_{\bar{n}n}\,\delta_{\bar{m}m}\;\; 2A_{nm}\,\sinh(\partial_{nm} z)$$

$$= \quad \frac{a^2}{2}\, A_{\bar{n}\bar{m}}\,\sinh(\partial_{\bar{n}\bar{m}} z)$$

again

$$\underbrace{\frac{a^2}{2} A_{nm}\,\sinh(\partial_{nm} z)}_{} = \int_0^a dx\,dy\;\; \sin\frac{\pi n}{a}x\;\; \sin\frac{\pi m}{a}y\;\; \phi(x,y,z)$$

coefficient of
$\sin\frac{\pi n}{a}x\ \sin\frac{\pi m}{a}y$

evaluating at $z = d/2$

$$\frac{a^2}{2} A_{nm}\,\sinh\left(\partial_{nm}\frac{d}{2}\right) = \int_0^a dx \int_0^a dy\;\; \sin\frac{\pi n}{a}x\;\; \sin\frac{\pi m}{a}y \cdot \frac{V}{2}$$

we need:

$$\int_0^a dx\;\; \sin\frac{\pi n}{a}x = \frac{a}{\pi n}\int_0^{n\pi} d\xi\,\sin\xi = \frac{a}{\pi n}\left[-\cos\xi\right]\Big|_0^{n\pi}$$

$$= \begin{cases} \dfrac{2a}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

then

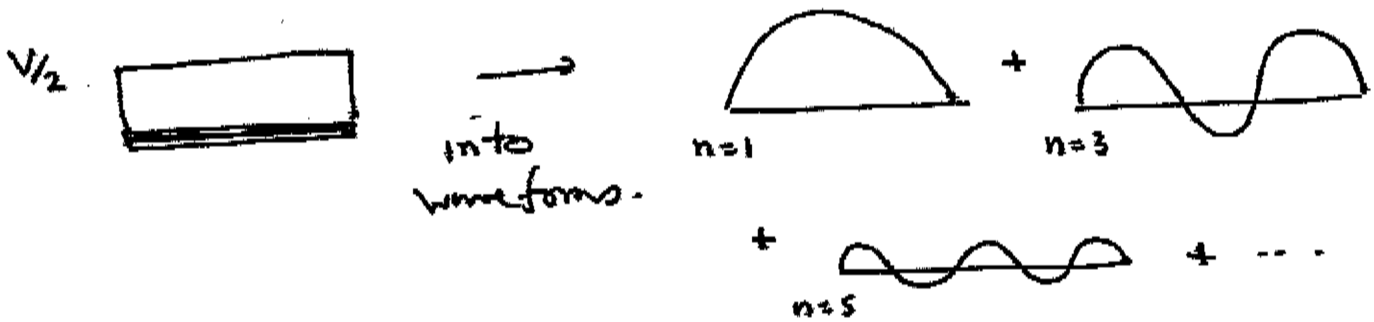
$$\frac{a^2}{2} A_{nm}\,\sinh\left(\partial_{nm}\frac{d}{2}\right) = \begin{cases} \dfrac{V}{2}\cdot\dfrac{4a^2}{\pi^2}\,\dfrac{1}{nm} & n, m \text{ } \underline{\text{both}} \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

a
$$A_{nm} = \frac{4V}{\pi^2}\,\frac{1}{nm}\,\frac{1}{\sinh(\lambda_{nm} d/2)} \qquad n,m \text{ odd}$$

so that

$$\phi(x,y,z) = \sum_{\substack{n,m=1\\ \text{odd}}}^{\infty} \frac{8}{\pi^2}V\,\frac{1}{nm}\,\frac{\sinh(\lambda_{nm} z)}{\sinh(\lambda_{nm} d/2)}\,\sin\frac{\pi n}{a}x\,\sin\frac{\pi m}{a}y$$

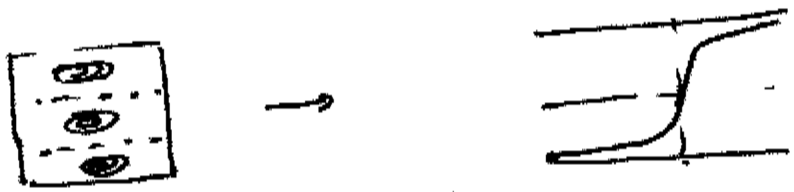
Let's go over again in words how we have solved this problem.
We have decomposed the potential on the boundary



(amplitude $\sim \frac{1}{n}$)

Each waveform gives a characteristic variation in z:

n = m = 1

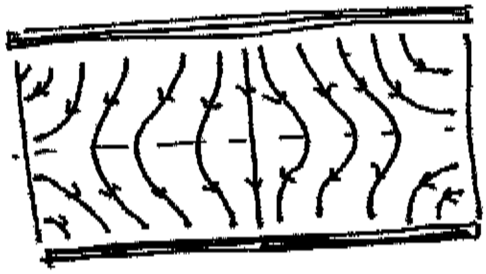


n=3 m=1

→ steeper and steeper z-dependence as $(n^2+m^2)^{1/2}$ increases.

The E field is strongest near the conductors and dissipates as we go toward the center

$\phi = V/2$



$\phi = 0$ $\phi = 0$

$\phi = -V/2$

What is the capacitance of this capacitor? The potential difference is V, and we can compute the charge Q from $\vec{E}(z = d/2)$:

$$E^z = -\frac{\partial}{\partial z}\phi(x,y,z)$$

$$= \sum_{\substack{n,m \\ odd}} -\frac{8}{\pi^2} V \frac{1}{nm} \overbrace{\frac{\pi}{a}(n^2+m^2)^{1/2}}^{\lambda_{nm}} \frac{\cosh(\lambda_{nm} d/2)}{\sinh(\lambda_{nm} d/2)} \sin\frac{\pi n}{a}x \sin\frac{\pi m}{a}y$$

$$= \sum_{\substack{n,m \\ odd}} -V \cdot \frac{8}{\pi a} \frac{(n^2+m^2)^{1/2}}{nm} \frac{\cosh \lambda_{nm} d/2}{\sinh \lambda_{nm} d/2} \sin\frac{\pi n}{a}x \sin\frac{\pi m}{a}y$$

$$\rho = \epsilon_0 \hat{n}\cdot\vec{E} \quad \text{so}$$

$$\rho = \sum_{\substack{nm \\ odd}} V \cdot \frac{8\epsilon_0}{\pi a} \frac{(n^2+m^2)^{1/2}}{nm} \frac{\cosh \lambda_{nm} d/2}{\sinh \lambda_{nm} d/2} \sin\frac{\pi n}{a}x \sin\frac{\pi m}{a}y$$

$$Q = \int_0^a dx\,dy\ \rho(x,y)$$

$$= \sum_{\substack{n,m\\ \text{odd}}} V \cdot \frac{8\epsilon_0}{\pi a} \frac{(n^2+m^2)^{\frac{1}{2}}}{nm} \frac{\cosh \lambda_{mn} d/2}{\sinh \lambda_{nm} d/2} \frac{2a}{\pi n} \frac{2a}{\pi m}$$

$$C = \frac{Q}{V} = \sum_{n,m\ \text{odd}} \frac{32\epsilon_0 a}{\pi^3} \frac{(n^2+m^2)^{\frac{1}{2}}}{n^2 m^2} \left(\frac{\cosh \lambda_{mn} d/2}{\sinh \lambda_{mn} d/2} \right)$$

To see better what is going on, consider the limit $d/a \to 0$

$$\cosh \lambda_{mn} d/2 \to 1 \qquad \sinh \lambda_{mn} d/2 \sim \lambda_{mn} d/2 = \frac{\pi}{2a} d\,[n^2+m^2]^{\frac{1}{2}}$$

$$C = \frac{Q}{V} \longrightarrow \sum_{\substack{n,m\\ \text{odd}}} \frac{64\epsilon_0}{\pi^4} \frac{a^2}{d} \frac{1}{n^2 m^2}$$

$$= \Big[\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{8}{\pi^2} \frac{1}{n^2} \Big]^2 \cdot \frac{a^2 \epsilon_0}{d}$$

To finish this, it would be good to know the value of

$$\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{1}{n^2}$$

My favorite place to look such things up is

Abramowitz + Stegun, Handbook of Mathematical Functions

Indeed, p. 808, 23.2.28: $\quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \pi^2/8 = 1.2337$

more generally,

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^s} = \left(1 - \frac{1}{2^s}\right) \zeta(s) \qquad \zeta = \text{Riemann zeta function}$$

so in the limit $d/a \to 0$

$$C = Q/V = \frac{\epsilon_0 a^2}{d}$$ and we recover the result of p.3

The formula on the previous page gives C for any ratio d/a.