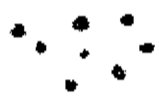
Laplace's Equation - 2

The equations of electrostatics are partial differential equations, and so we must ask what boundary conditions the solutions satisfy. So far, we have only considered the electric fields of fixed charges:



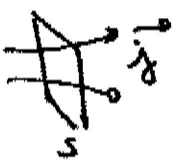
For which we solve $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ $\vec{\nabla}\times\vec{E} = 0$ subject to $\vec{E}\to 0$ as $|\vec{r}|\to\infty$. There are interesting situations, however, in which we don't know ρ, and in which the determination of ρ involves computing the $\vec{E}$ fields. If we put a material into an electric field, charges will move in the material under the influence of the field

If the material is a dielectric, its atoms will polarize. If the material is a conductor, electric currents will flow. In either case, the distribution of charge in the material needs to be determined together with the $\vec{E}$ field.

We'll discuss dielectrics later in the course. Now, however, I would like to give an idealized view of the behavior of a conductor which will allow us to solve electrostatic problems involving conductors and free charges.

A conductor is a material in which electric charge currents can flow. What actually flows is typically electrons (i.e. negatively charged particles), but we represent the flow of charge as a current pointing in the direction of positive charge flow

$$\vec{j} = C/m^2\,sec$$



s.t. $\int_S d^2x\, \hat{n}\cdot\vec{j}$ = C/sec flowing through S

Electric charge is conserved, and so $\vec{j}$ obeys a continuity equation

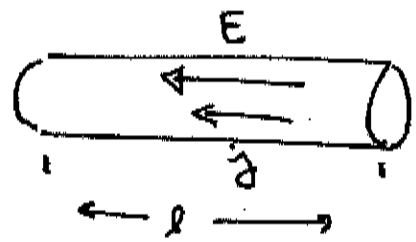
$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0 \qquad \rho = \text{charge density} \quad C/m^3$$

If there is an electric field inside a conductor, it will drive current flow. Typically, the resulting current is approximately linearly proportional to the field:

$$\vec{j} = \sigma\,\vec{E}$$

σ is called the conductivity. The $\vec{E}$ field exerts a force on the charges, but these charges are also slowed down by restraining forces. In a metal, electrons are slowed when they hit defects or impurities. In an ionic liquid, ions are slowed when they hit molecules of the solvent. In either case, the charge carriers reach a terminal velocity at which the force of $\vec{E}$ balances the restraining forces.

If you think about the flow of charge in a wire:



the flux of $\vec{J}$ out of the wire

$$\int d^2x\, \hat{n}\cdot\vec{J} = I \quad C/\text{sec} = \text{Ampere}$$

the current carried by the wire.

If $\vec{J} \propto \vec{E}$ also $I \propto \vec{E}$. The potential drop across the wire is

$$\Delta\phi = E\,l$$

so we can also write

$$I = \Sigma\, \Delta\phi \qquad \Sigma = \text{conductivity}$$

In this case, we normally write $\Delta\phi \equiv V$, measure $\Delta\phi$ in the units

$$N\text{-}m/C = J/C = \underline{\text{Volts}}$$

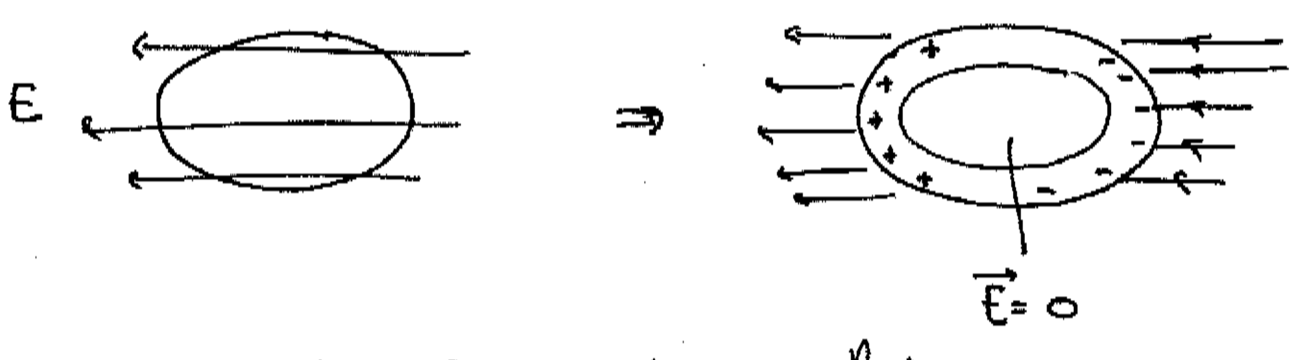
and call $V = \Delta\phi$ the "Voltage". The relation between I and V is then written

$$V = IR$$

Ohm's law $\quad R$ = resistance in $\frac{\text{Volt}}{\text{Ampere}}$.

The ~~underlying~~ (still phenomenological) relation $\vec{j} = \sigma \vec{E}$ is also called Ohm's law.

In any conductor, it takes a finite time for the currents to flow in response to $\vec{E}$. Eventually, however, they are done, and charge moves to cancel $\vec{E}$ out.



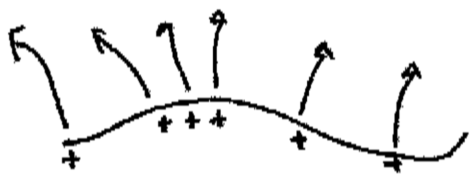
The static situation for a conductor is that $\vec{E} = 0$ in its interior; otherwise charges would still be flowing. When we use the idealization of a perfect conductor, we assume that we have waited long enough for the flow of current to finish.

If $\vec{E} = 0$ in the interior of a conductor, the charges that cancel $\vec{E}$ must be located exactly on the surface of the conductor, in a surface layer. Actually, this is not such a bad approximation, at least for a metal. Remember that 1 electron carries

$$|e| = 1.6 \times 10^{-19}\,C \quad \text{so} \quad 1\,C = 6 \times 10^{18}\,e.$$

But $1\,m^2 = (10^{10}\,\text{Å})^2 \simeq 10^{20}$ atoms. So there is room for the electrons in a surface layer 1 atom thick.

$\vec{E} = 0 \Rightarrow \phi = \underline{\text{constant}}$ inside a conductor. At the surface ϕ can vary into free space, but ϕ must be constant along the surface. So the typical situation is:

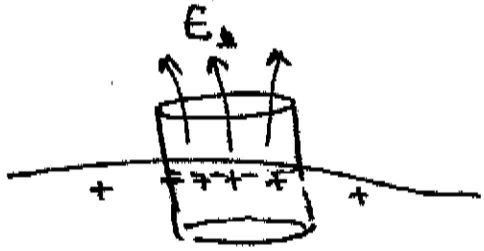


But, since ϕ is constant in the conductor

ϕ = constant on the surface, $\quad \vec{\nabla}\phi \parallel \hat{n}$ normal to the surface

$\vec{E} \parallel \hat{n}$

If the surface density of charge is σ C/m^2, we can determine this from Gauss' law. Consider the small cylinder



with sides normal to the surface.

the flux of $\vec{E}$ thigh this surface is

$$\Phi = \int_{\text{top}} d^2x\, \hat{n}\cdot E + \int_{\text{bottom}} d^2x\, \hat{n}\cdot E = (\text{Area})\cdot|\vec{E}|$$

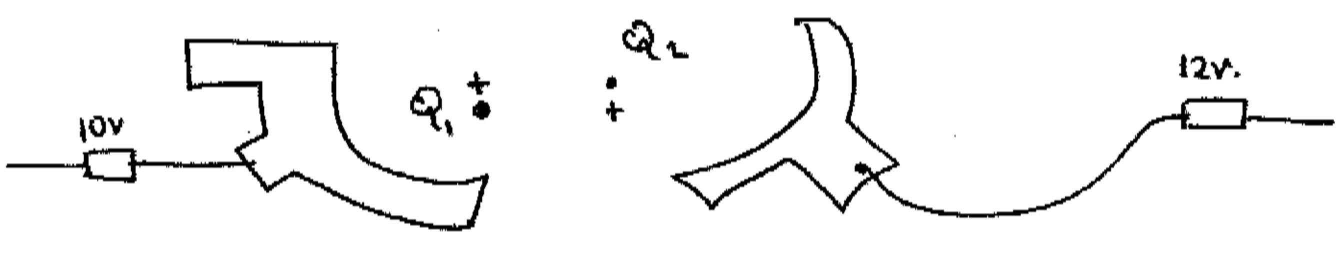
$$= \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{1}{\epsilon_0}(\text{Area})\cdot\sigma$$

so the electric field just outside the conductor is

σ C/m^2

$$\vec{E} = \frac{\sigma}{\epsilon_0}\cdot\hat{n}$$

Now we can set up a typical electrostatic problem: We have some charges $Q_1 - Q_N$. We also have some conductors, of arbitrary shape. We might know the potential on each conductor (e.g. because the conductor is connected to a battery that sets up a fixed potential difference from a common site ("ground")) or we might know the surface charge density:



the problem is the find the $\vec{E}$ field and potential $\phi(\vec{x})$ in such a situation.

Here is an approach to this problem. Represent $\vec{E}$ by

$$\vec{E} = -\vec{\nabla}\phi$$

Putting this into $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$, we have.

$$-\nabla^2\phi = \frac{1}{\epsilon_0}\rho \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We put the known charges into $\rho(\vec{x})$. Then we must solve this equation in the domain outside the conductors, with the boundary condition that ϕ is fixed or $\hat{n}\cdot\vec{\nabla}\phi = -\frac{\rho}{\epsilon_0}$ is fixed on the boundaries. The equation

$$-\nabla^2\phi = \frac{1}{\epsilon_0}\rho$$

is called Poisson's equation. With $\rho = 0$,

$$-\nabla^2\phi = 0$$

this is Laplace's equation. The operator ∇^2 is called the Laplacian.

I would now like to prove that the conditions I have listed uniquely determine ϕ. Some physicists consider "existence and uniqueness theorems" purely mathematics. But I think that it is very important to know when there is a theorem of this type. If a set of conditions ensure that Poisson's equation has a unique solution, then you are free to try to find a solution by any method, however bizarre. If you succeed, that solution is the only one.

Here is the proof: We are looking for solutions of

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \rho$$

in a volume V, s.t. on the surface $S = \partial V$ we specify either the value of ϕ or the normal derivative $\hat{n} \cdot \vec{\nabla} \phi$. Assume that there are two solutions meeting these conditions: $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$. Then

$$-\nabla^2(\phi_1 - \phi_2) = 0 \quad \text{in the interior of } V$$

and either $\phi_1 - \phi_2$ or $\hat{n} \cdot \vec{\nabla}(\phi_1 - \phi_2)$ vanishes on the boundary. Consider the quantity

$$A = \int_V d^3x \, \left[\vec{\nabla}(\phi_1 - \phi_2)\right]^2$$

manifestly $A \geq 0$, and $A = 0$ only if $\phi_1 = \phi_2$ everywhere. [or, if we specify only $\hat{n} \cdot \vec{\nabla}(\phi_1 - \phi_2)$, if $\phi_1 - \phi_2 = \text{const}$].

But

$$A = \int_V d^3x \, \left[\vec{\nabla} \left\{ (\phi_1 - \phi_2) \, \vec{\nabla}(\phi_1 - \phi_2) \right\} \; - \; (\phi_1 - \phi_2) \, \nabla^2 (\phi_1 - \phi_2) \right]$$

the second term vanishes because $\nabla^2(\phi_1 - \phi_2) = 0$. The first term is a total divergence, and so

$$= \int_{S = \partial V} d^2x \;\; \hat{n} \cdot [\vec{\nabla}(\phi_1 - \phi_2)] \cdot (\phi_1 - \phi_2)$$

Since at each point on the boundary $\phi_1 = \phi_2$ or $\hat{n} \cdot \vec{\nabla} \phi_1 = \hat{n} \cdot \vec{\nabla} \phi_2$

$$= 0$$

so ϕ_1 and ϕ_2 are identical !

Boundary conditions in which ϕ is specified are called Dirichlet boundary conditions. Boundary conditions in which $\hat{n} \cdot \vec{\nabla} \phi$ are specified are called Neumann boundary conditions. Either type of boundary condition picks out a unique solution to Laplace's or Poisson's equation.

A small variation on this argument gives a little more information about the solution of Laplace's equation with Dirichlet boundary conditions. Let $\phi(\vec{x})$ be this solution, and let $\chi(\vec{x})$ be any other function satisfying the same boundary conditions.

Consider the integral

$$\mathcal{E} = \frac{1}{2} \int_V d^3x \, (\vec{\nabla}\chi)^2$$

which equals the total energy if χ is an electrostatic potential

Write $\chi = \phi + \alpha(\vec{x})$; by the boundary conditions, $\alpha = 0$ on $S = \partial V$

$$\mathcal{E} = \int_V d^3x \left\{ \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \vec{\nabla}\phi \cdot \vec{\nabla}\alpha + \tfrac{1}{2}(\vec{\nabla}\alpha)^2 \right\}$$

consider the middle term more closely; this equals

$$\int_V d^3x \left\{ \vec{\nabla}(\vec{\nabla}\phi \cdot \alpha) - \nabla^2\phi \cdot \alpha \right\} \qquad \text{but } \nabla^2\phi = 0$$

$$= \int_{S=\partial V} d^2x \; \hat{n}\cdot\vec{\nabla}\phi \, \alpha = 0 \quad \text{since } \alpha = 0 \text{ on the boundary}$$

so

$$\mathcal{E} = \int_V d^3x \left[\tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}(\vec{\nabla}\alpha)^2 \right] > \int d^3x \tfrac{1}{2}(\vec{\nabla}\phi)^2$$

if $\alpha \neq 0$.

So, ϕ is the electrostatic potential that minimizes the energy integral subject to Dirichlet boundary conditions.

From these considerations, you might visualize that the solution of Laplace's equation is the smoothest ~~[illegible]~~ function that interpolates the boundary conditions. I'd like to consider one more formal property that confirms this impression. Let $\phi(\vec{x})$ be a solution of Laplace's equation. Taylor expand it about any point $\vec{x}_0$

$$\phi(\vec{x}) = \phi(\vec{x}_0) + (x-x_0)^i \psi_i + \tfrac{1}{2}(x-x_0)^i (x-x_0)^j \theta_{ij} + \cdots$$

where ψ_i, θ_{ij} are constants given by

$$\psi_i = \frac{\partial}{\partial x^i}\phi\Big|_{x_0} \qquad \theta_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}\theta\Big|_{x_0}$$

Laplace's equation says that

$$\theta_{11} + \theta_{22} + \theta_{33} = 0$$

If x_0 were a maximum of ϕ, we would have $\psi_i = 0$ and θ_{11}, θ_{22}, θ_{33} all < 0. Laplace's equation says this is impossible. Similarly, at a minimum of ϕ, $\psi_i = 0$ and $\theta_{11}, \theta_{22}, \theta_{33} > 0$. Again, impossible. A solution of Laplace's equation in V cannot have local maxima or minima in V. (It can have "saddle points", where $\psi_i = 0$ and two of θ_{11} θ_{22} θ_{33} are > 0 or < 0.)

There is an even stronger statement. If we average over a small sphere of radius a centered on x_0

$$\langle (x-x_0)^i \rangle = 0 \qquad \text{by symmetry}$$

$$\langle (x-x_0)^i (x-x_0)^j \rangle = c\, a^2\, \delta^{ij} \qquad \text{by spherical symmetry}$$

[Here are two evaluation of c:

$i,j=z$ $$\langle (x-x_0)^z (x-x_0)^z \rangle = \int \frac{d\theta \sin\theta \, d\phi}{4\pi} \; a^2 \cos^2\theta = \frac{a^2}{3}$$

$i,j=x$ $$\langle (x-x_0)^x (x-x_0)^x \rangle = \int \frac{d\theta \sin\theta \, d\phi}{4\pi} \; (a \sin\theta \cos\phi)^2 = a^2 \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{a^2}{3}$$

so $c = \frac{1}{3}$.]

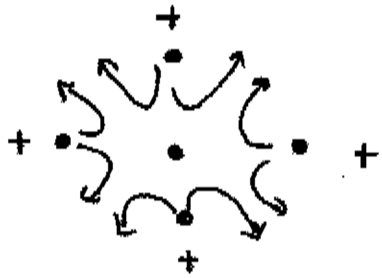
$$\langle \phi(\vec{x}) \rangle = \phi(\vec{x}_0) + 0 + \frac{1}{2} \cdot \frac{a^2}{3} \cdot \underbrace{(\Theta_{11} + \Theta_{22} + \Theta_{33})}_{=0} + \cdots$$

so for a small sphere, the value of ϕ at the center is the average of ϕ over the sphere. Griffiths gives an explicit calculation that shows this is true even for a finite sphere.

The fact that ϕ cannot have a local minimum implies

<u>Earnshaw's theorem</u>: A charged particle cannot be held in stable equilibrium by electrostatic forces. Proof: the particle can always run <u>down</u> to a lower value of its potential energy. There are circumstances where no net electrostatic force acts

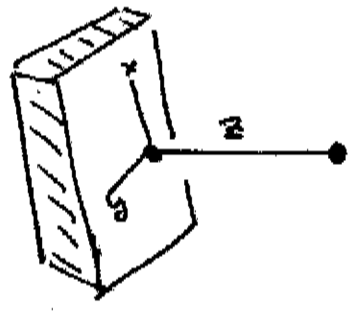
on a particle eg.



But, if the charge at the center is moved slightly, it will run off to one of the + charges, if it is negative, or out of the paper, if it is positive.

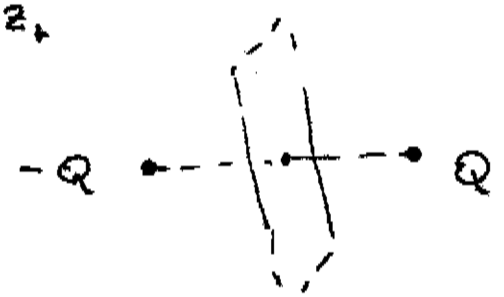
Now let's look at some solutions to the Laplace and Poisson equations. In the next few lectures, we will consider systematic methods for solving these equations, but I'd like to start with a totally unsystematic method.

Consider the problem of a charge Q brought near a perfectly conducting plane:



I take the surface of the conductor to be the xy plane, and the charge to be at a distance z. I assumed that the conductor is "grounded", that is, connected to a reservoir of charge at $\phi = 0$. What is the solution of the Poisson equation ?

Consider the following completely different problem: Put a charge Q at $\underset{\overset{\shortparallel}{z_+}}{(0,0,z)}$ and a charge $-Q$ at $\underset{\overset{\shortparallel}{z_-}}{(0,0,-z)}$.



For this situation the potential is

$$\phi = \frac{Q}{4\pi\epsilon_0 |\vec{r}-\vec{z}_+|} - \frac{Q}{4\pi\epsilon_0 |\vec{r}-\vec{z}_-|}$$

this potential satisfies

$$-\nabla^2\phi = \frac{Q}{\epsilon_0}\left[\delta^{(3)}(\vec{r}-\vec{z}_+) - \delta^{(3)}(\vec{r}-\vec{z}_-)\right]$$

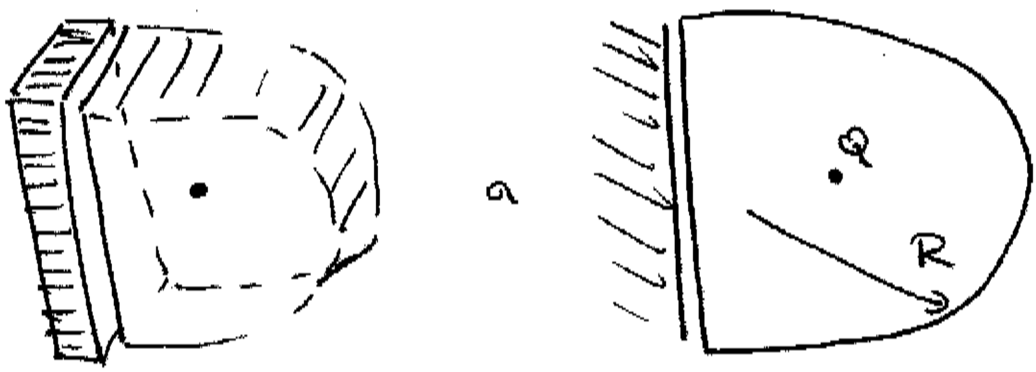
so in the region $z>0$

$$-\nabla^2\phi = \frac{Q}{\epsilon_0}\,\delta^{(3)}(\vec{r}-\vec{z}_+)$$

on the x,y plane $|\vec{r}-\vec{z}_+| = |\vec{r}-\vec{z}_-|$, so $\phi = 0$

So this ϕ satisfies the correct Poisson equation, with the correct boundary condition. Thus it is the unique solution to our original problem. The charge $-Q$ is completely fictitious, a mathematical trick. But that doesn't matter. The charge at $-Q$ is called an "image charge", and the trick is the "method of images".

The charge $-Q$ actually has a sort of reality. Note that, for $|\vec{r}| \gg z$, $\phi(\vec{r}) \sim \frac{1}{r^2}$, so $\vec{E} \sim \frac{1}{r^3}$. Now consider the flux of E through the surface:



The outside surface has a flux $\Phi = \int d^2x\, \hat{n}\cdot E \sim R^2 \frac{1}{R^3} \to 0$ as $R \to \infty$. The flux through the piece of the surface which lies against the conducting plane is

$$\Phi = \int d^2x\, \hat{n}\cdot E = \int d^2x \frac{(-\rho)}{\epsilon_0} \quad \text{where } \rho \text{ is}$$

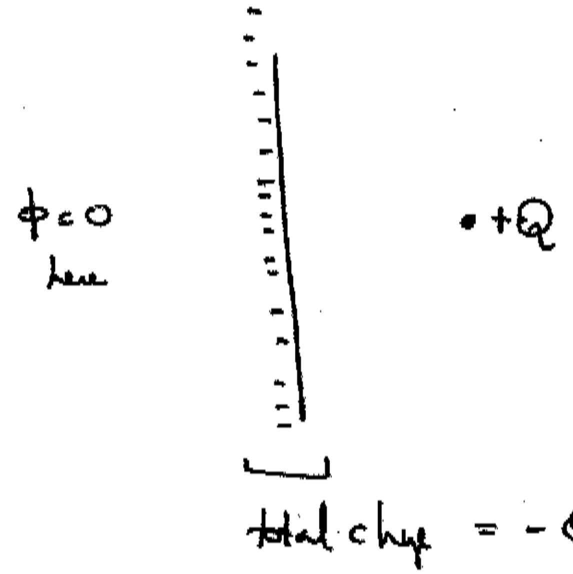
the surface charge density. On the other hand, by Gauss' law,

$$\Phi = \frac{Q}{\epsilon_0}$$

so

$$-Q = \int d^2x\, \rho$$

that is, the chge Q draws up from ground a total charge of exactly $-Q$, which is distributed in a surface layer:



We can compute the explicit distributn of this chge from the relatn $\rho = \epsilon_0 E_\perp$

$$E = -\vec{\nabla}\phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{\widehat{r-z_+}}{|r-z_+|^2} - \frac{\widehat{r-z_-}}{|r-z_-|^2}\right)$$

$$\underset{\text{at } r=(x,y,0)}{=} \frac{Q}{4\pi\epsilon_0}\left(0,\ 0,\ \frac{-2z}{[x^2+y^2+z^2]^{3/2}}\right)$$

so

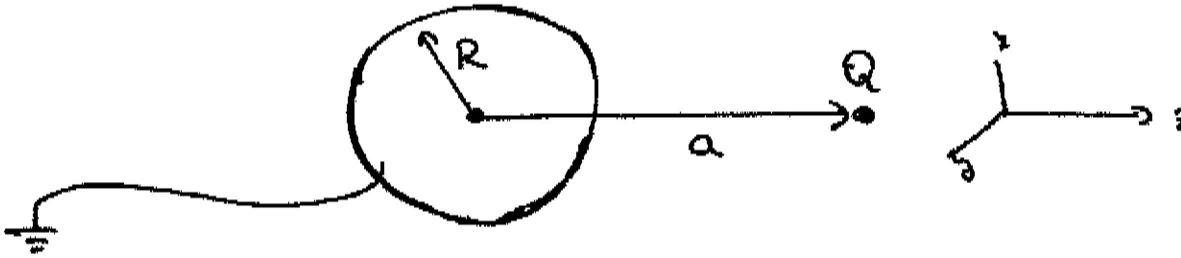
$$\rho = -\frac{Q}{2\pi}\,\frac{z}{[x^2+y^2+z^2]^{3/2}}$$

The chge $+Q$ is also attracted to the conductg plane by a force

$$\vec{F} = -\hat{z}\,\frac{Q^2}{4\pi\epsilon_0}\frac{1}{(2z)^2} = -\hat{z}\,\frac{Q^2}{16\pi\epsilon_0 z^2}$$

which we might attribute to the image charge.

There is another, even more surprising, problem that can be solved by the method of images. Consider a conducting sphere of radius R and a charge Q at a distance a from the center of the sphere, on the $\hat{z}$ axis



The sphere is a surface with $\phi =$ const.; I'll assume that it is connected to a reservoir of charge at $\phi = 0$. What are the $\vec{E}$ fields in the space outside the sphere?

Consider what would be the potential if we took the charge Q at $\underbrace{(0,0,a)}_{\vec{z}_a}$ and added a charge

$$-Q' = -\frac{R}{a}Q \quad \text{at} \quad \underbrace{(0,0,b)}_{\vec{z}_b} \qquad b = R^2/a.$$

We would have

$$\phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\vec{z}_a|} - \frac{R}{a}\frac{1}{|\vec{r}-\vec{z}_b|}\right)$$

At the two points where the sphere of radius R meets the z axis.

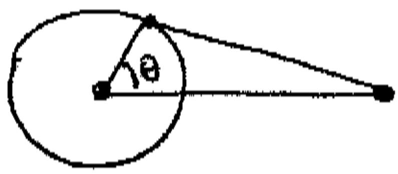
$\vec{r} = (0\ 0\ R)$ $\qquad \phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|a-R|} - \frac{R}{a}\frac{1}{|R - R^2/a|}\right)$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|a-R|} - \frac{1}{|a-R|}\right) = 0$$

$\vec{r} = (0,0,-R)$ $\qquad \phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|a+R|} - \frac{R}{a}\frac{1}{|R+R^2/a|}\right) = 0$

Actually, $\phi = 0$ at all points on this sphere. Consider the point

$\vec{r} = (R\sin\theta, 0, R\cos\theta)$



$$\phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{[(a-R\cos\theta)^2 + (R\sin\theta)^2]^{1/2}} - \frac{R}{a}\frac{1}{[(R^2/a - R\cos\theta)^2 + (R\sin\theta)^2]^{1/2}}\right)$$

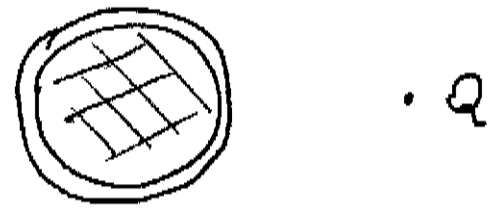
$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{[a^2 - 2Ra\cos\theta + R^2]^{1/2}} - \frac{R}{a}\frac{1}{[R^4/a^2 - 2R^3/a\cos\theta + R^2]^{1/2}}\right)$$

$$= 0 \ !$$

So this potential is the unique solution to the problem we originally posed.

The charge pulled onto the sphere can be computed from Gauss' law:

for the surface that just wraps the sphere:



the flux is just that of the image charge:

$$\int d^3x\, \rho = -\frac{R}{a} Q$$

This charge is distributed over the sphere 

and $\Phi = 0$ $\vec{E} = 0$ in the interior of the sphere. The force on the charge Q is

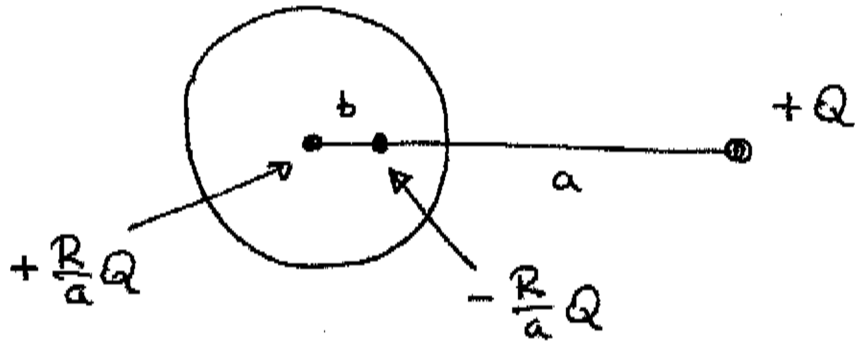
$$\vec{F} = -Q\frac{R}{a} Q \frac{1}{4\pi\epsilon_0} \frac{\hat{z}}{(z_a - z_b)^2}$$

$$= -\frac{Q^2}{4\pi\epsilon_0}\hat{z}\,\frac{R}{a}\,\frac{1}{(a - R^2/a)^2}$$

$$\vec{F} = -\frac{Q^2}{4\pi\epsilon_0}\hat{z}\,\frac{aR}{(a^2 - R^2)^2}$$

Now, what if the conducting sphere was not grounded but just sitting isolated in free space, and contained zero net charge before we brought in the charge Q. As we

moved Q to $z = z_a$, the charge on the conducting sphere must remain zero, and the sphere at $|\vec{r}| = R$ must remain an equipotential, though not necessarily at $\phi = 0$.
We can solve these conditions by adding another image charge at $\vec{r} = 0$

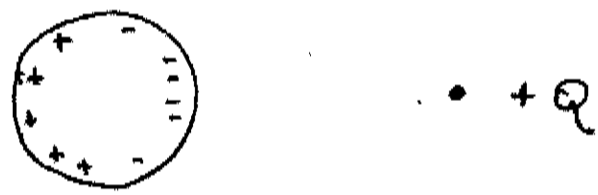


$$\phi = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec{r}-\vec{z}_a|} - \frac{R}{a}\frac{1}{|\vec{r}-\vec{z}_b|} + \frac{R}{a}\frac{1}{|\vec{r}|}\right\}$$

and this has the constant value

$$\phi = \frac{Q}{4\pi\epsilon_0 a} \qquad \text{at} \quad |\vec{r}| = R$$

(relative to $\phi = 0$ at $|\vec{r}| = \infty$). The surface charge on the sphere takes both positive and negative values:



I'd like to do one more trick with this configuration.

Consider the limit $a \to \infty$, R fixed. Let $\vec{r} = (x, y, z)$

$r = (x^2+y^2+z^2)^{1/2}$

$$|r - z_a|^{-1} = [(a-z)^2 + x^2 + y^2]^{-1/2}$$

$$= [a^2 - 2az + z^2 + x^2 + y^2]^{-1/2}$$

$$= \frac{1}{a}\left[1 - 2\frac{z}{a} + \frac{z^2+x^2+y^2}{a^2}\right]^{-1/2}$$

$$= \frac{1}{a}\left[1 + \frac{z}{a} + \left(\text{terms w. } \frac{x^2, y^2, z^2}{a^2}\right) + \cdots\right]$$

$$|r - z_b|^{-1} = [(z - R^2/a)^2 + x^2 + y^2]^{-1/2}$$

$$= \left[(x^2+y^2+z^2) - 2\frac{R^2 z}{a} + \cdots\right]^{-1/2}$$

$$= \frac{1}{r}\left[1 - 2\frac{R^2 z}{a r^2} + \cdots\right]^{-1/2}$$

$$= \frac{1}{r} + \frac{R^2 z}{a r^3} + \cdots$$

then

$$\phi = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{a} + \frac{z}{a^2} + \cdots - \frac{R}{a}\left[\cancel{\frac{1}{r}} + \frac{R^2 z}{a r^3} + \cdots\right] + \cancel{\frac{R}{a}\frac{1}{r}}\right\}$$

$$= \left(\underset{x,y,z}{\text{constant, indep of}}\right) + \frac{Q}{4\pi\epsilon_0 a^2}\left(z - \frac{R^3 z}{r^3}\right)$$

$$+ \left(\text{terms} \propto \frac{Q}{a^3}\cdot(x^2, y^2, z^2)\right)$$

send $a \to \infty$, $Q \to \infty$ s.t.

$$\frac{Q'}{4\pi\epsilon_0 a^2} = \text{fixed , call this combination} \equiv E_0$$

dropping the constant , we obtain

$$\phi = E_0 \left(z - \frac{R^3 z}{r^3} \right)$$

You can check that this function

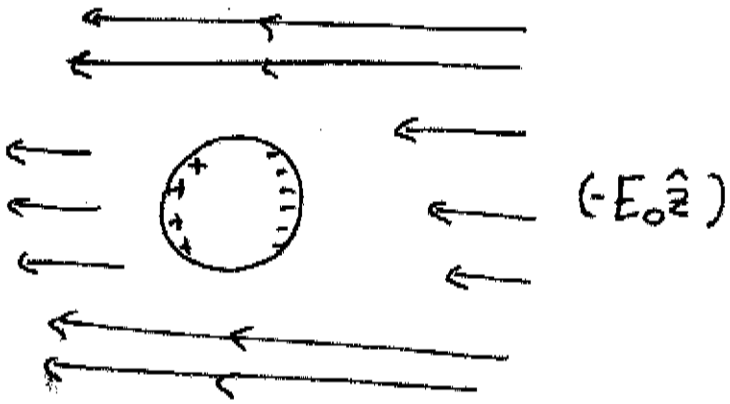
(1) solves Laplace's equation for $|\vec{r}| > R$

(2) satisfies $\phi = 0$ on the surface $|\vec{r}| = R$

What is the interpretation of this solution?

At large $|\vec{r}|$, $\vec{E} = -\vec{\nabla}\phi \approx -E_0 \hat{z}$.

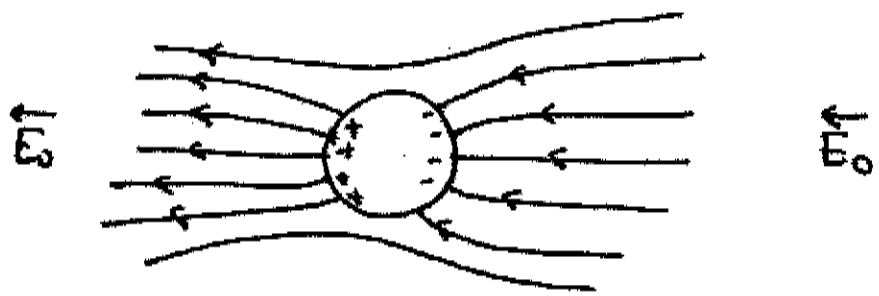
So this is the solution of Laplace's equation appropriate for a neutral conducting sphere of radius R immersed in a constant $\vec{E}$ field



more exactly, $\vec{E} = -\vec{\nabla}\phi$

$$E^x = -E_0 \frac{3xzR^3}{r^5} \qquad E^y = -E_0 \frac{3yzR^3}{r^5}$$

$$E^z = -E_0 \left(1 - \frac{R^3}{r^5}[r^2 - 3z^2]\right)$$



At a point on the sphere $\vec{r} = (R\sin\theta, 0, R\cos\theta)$, for example,

$$E^x = -E_0 \cdot 3\sin\theta\cos\theta \qquad E^y = 0$$

$$E^z = -E_0 \cdot 3\cos^2\theta$$

so

$$\vec{E} = -\hat{n}\, E_0 \cdot 3\cos\theta$$

where $\hat{n} = (\sin\theta, 0, \cos\theta)$ is the outward normal to the sphere. $\vec{E}$ is normal to the conducting sphere, as it should be, and we can read off the surface charge

$$\rho = -(3\underline{E_0\cos\theta}) \cdot \epsilon_0$$