Oct. 4

Laplace's Equation - 1

Go back to the formula for an isolated point charge

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \qquad r = [x^2+y^2+z^2]^{\frac{1}{2}}$$

and now take the curl. A typical derivative that we need is

$$\frac{\partial}{\partial x}E^y = \frac{\partial}{\partial x}\left[\frac{1}{4\pi\epsilon_0}Q\left(\frac{y}{[x^2+y^2+z^2]^{3/2}}\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0}Q\left(-\frac{3}{2}\cdot 2\right)\frac{xy}{[x^2+y^2+z^2]^{5/2}}$$

$$= -\frac{3Q}{4\pi\epsilon_0}\frac{xy}{r^5}$$

so

$$\frac{\partial}{\partial x}E^y - \frac{\partial}{\partial y}E^x = 0$$ and similarly for other components of the curl.

$$\vec{\nabla}\times\vec{E} = 0$$

In fact, there is no large loop on which $\oint d\vec{x}\cdot\vec{E} = 0$, even a loop that goes right into the point charge:

$$\vec{\nabla}\times\vec{E} = 0 \quad \text{w. no. source}$$

By superposition, this equation holds also for any collection of charges. So now we have a more complete set of equations for electrostatics

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = 0$$

It looks like this is 4 equations for 3 unknowns $E^x\ E^y\ E^z$ at $\vec{r}$. Actually, there is one constraint among these equations, as I'll show in a moment. So we have 3 equations in 3 unknowns, just the right number.

How do we solve these equations? We can simplify them, at least, by doing a little more of the math of the operator $\vec{\nabla}$. Let $t(\vec{r})$ be a scalar function. Then

$$\vec{\nabla}t$$

is a vector field. I claim that $\vec{\nabla}\times(\vec{\nabla}t) = 0$.

We can see this by direct computation:

$$[\vec{\nabla}\times\vec{\nabla}t]^z = \frac{\partial}{\partial x}\frac{\partial t}{\partial y} - \frac{\partial}{\partial y}\frac{\partial t}{\partial x} = 0$$

or, more automatically,

$$[\vec{\nabla}\times(\vec{\nabla}t)]^i = \epsilon^{ijk}\frac{\partial}{\partial x^j}(\vec{\nabla}t)^k = \epsilon^{ijk}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k}t = 0$$

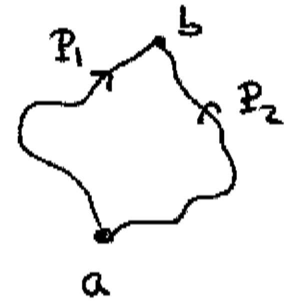
by the antisymmetry of ϵ^{ijk} ! The converse of this relation is also true. Let $\vec{\omega}$ be a vector field s.t. $\vec{\nabla}\times\vec{\omega} = 0$ throughout a contractible (topologically trivial) region R. Then there exists $\theta(\vec{r})$, a scalar function, s.t.

$$\vec{\nabla}\theta(\vec{r}) = \vec{\omega}$$

It is not so hard to find θ. Let $\vec{a}$ be a point in R and define $\theta(\vec{a}) = 0$. Then define

$$\theta(\vec{b}) = \int_a^b d\vec{x}\cdot\vec{\omega}$$

There is a danger that this definition might be ambiguous, because in principle it could depend on the path by which we go from a to b. Let P_1 P_2 be two different paths. Our definition makes sense only if



$$0 = \int_{a\,P_1}^{b} d\vec{x}\cdot\vec{\omega} - \int_{a\,P_2}^{b} d\vec{x}\cdot\vec{\omega}$$

but

$$= \oint_{P_1-P_2} d\vec{x}\cdot\vec{\omega} = \int_S d^2x\, \hat{n}\cdot(\vec{\nabla}\times\vec{\omega}) = 0$$

where S is a surface spanning the closed curve, since $\vec{\nabla}\times\vec{\omega} = 0$!

Again, our definition of Θ is consistent only if $\vec{\nabla}\times\vec{\omega}=0$.

Notice that it was also important that there exist a surface S that spans the closed curve. This is true in a contractible space. It isn't true, for example, on a torus!

One of the major theorems of 20^{th} century mathematics, de Rham's theorem, relates the ambiguity in defining Θ to the nontrivial topology of R (and generalizes these notions to higher dimension).

There is a similar relation between the curl and the divergence. Let $\vec{\omega}$ be a vector field. I claim that

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\omega}) = 0$$

The proof is again easily given with ϵ^{ijk}:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\omega}) = \frac{\partial}{\partial x^i}\ \epsilon^{ijk}\frac{\partial}{\partial x^j}\omega^k = 0$$

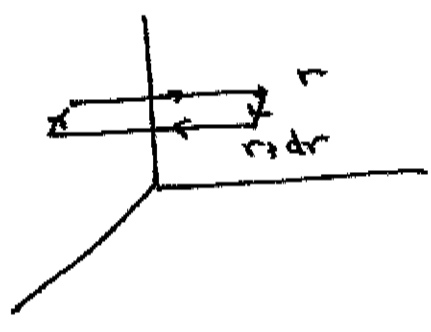
by antisymmetry of ϵ^{ijk}.

Now we can ask, if $\vec{\nabla}\cdot\vec{v}=0$ for a vector field $\vec{v}(\vec{r})$, is there a $\vec{\omega}$ s.t. $\vec{v}=\vec{\nabla}\times\vec{\omega}$?

To construct $\vec{\omega}$, set $\omega^x = 0$ everywhere and set $\vec{\omega} = 0$ on the y, z plane. Then compute $\vec{\omega}$ at $\vec{r}$ by using the relation

$$\oint d\vec{x}\,\vec{\omega} = \int d^2x\ \hat{n}\cdot\vec{\omega}$$

around the loop



for which

$$\oint d\vec{x}\cdot\vec{\omega} = \vec{\omega}\cdot d\vec{r}$$

$$\vec{v} = \vec{\nabla}\times\vec{\omega} \text{ if } \int_S d^2x\ \hat{n}\cdot\vec{v} = \int_{C=\partial S} d\vec{x}\cdot\vec{\omega}$$ for any surface

S. This is guaranteed because if two surfaces S_1 S_2 have the same boundary C

$$\int_{S_1} d^2x\ \hat{n}\cdot\vec{v} - \int_{S_2} d^2x\ \hat{n}\cdot\vec{v}$$

$$= \int_{\substack{V \\ \text{enclosed}}} d^3x\ \vec{\nabla}\cdot\vec{v} = 0$$

Again, there is no ambiguity only if V always exists i.e. only if R is topologically trivial.

$\vec{\omega}$ is not unique : $\vec{\omega}$ and $\vec{\omega} + \vec{\nabla}t$ have the same curl for any t.

By the way, since automatically $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\omega}) = 0$,
the equation $\vec{\nabla}\times\vec{E} = 0$ obeys a constraint and so
counts as 2 equations, not 3, as we noted above.

Since $\vec{\nabla}\times\vec{E} = 0$, we can represent E as

$$\vec{E} = -\vec{\nabla}\phi(\vec{r})$$

for some scalar function $\phi(\vec{r})$. ϕ is called the "electrostatic potential". (Griffiths writes $\vec{E} = -\vec{\nabla}V$.)

As a check, look back at Coulomb's law. For an isolated charge:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

notice that this is

$$\vec{E} = -\vec{\nabla}\left(\frac{Q}{4\pi\epsilon_0}\frac{1}{r}\right)$$

e.g., the x component is

$$-\frac{\partial}{\partial x}\left(\frac{Q}{4\pi\epsilon_0}\frac{1}{[x^2+y^2+z^2]^{1/2}}\right) = -\frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{2}\right)2\,\frac{x}{[x^2+y^2+z^2]^{3/2}}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{x}{r^3} \quad \checkmark$$

For a distribution of fixed charges:

$$\vec{E} = -\vec{\nabla}\phi \quad \text{where} \quad \phi = \sum_i \frac{Q_i}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}_i|}$$

This is an easier way to compute $\vec{E}$ than by directly summing fields; we sum or ~~integrate~~ over scalar functions ϕ and then take the gradient to find $\vec{E}$.

The relation between $\vec{E}$ and ϕ looks very similar to the relation between force $\vec{F}$ and potential energy V

$$\vec{F} = -\vec{\nabla}V$$

the work required to move a particle from a to b in a field of force is

$$W = \int_a^b d\vec{x}\cdot\vec{F}_{\text{applied}} = -\int_a^b d\vec{x}\cdot\vec{F}_{\text{restraining}}$$

In an electric field, a charge q (C) feels the force

$$\vec{F} = q\vec{E}$$

and so the work required to move it from a to b is

$$W = -\int_a^b d\vec{x}\cdot\vec{F} = -\int_a^b d\vec{x}\cdot q\vec{E} = \int_a^b d\vec{x}\; q\,\vec{\nabla}\phi$$

so the work required to move a charge q from a to b is

$$W = q\,[\,\phi(b) - \phi(a)\,]$$

The potential energy of a charge in an electrostatic field is then

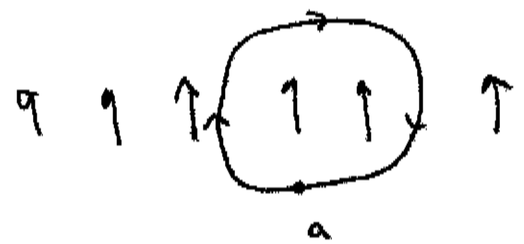
$$V(b) = q\,\phi(b) - (\text{const})$$

where (const) is the value of $q\phi$ at some reference point.

Often we take $\phi = 0$ at $r = \infty$, then

$$V(\vec{r}) = q\,\phi(\vec{r})$$

In mechanics, we can define a potential energy consistently if the net work required to move a particle from a along an arbitrary curve back to a is zero

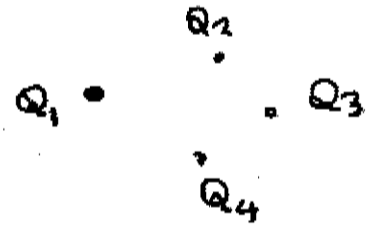


This is the condition $\oint d\vec{x}\cdot\vec{F} = 0$ or $\vec{\nabla}\times\vec{F} = 0$, which is satisfied by the electrostatic field of force

$$\vec{F} = q\,\vec{E} \quad \text{w.} \quad \vec{\nabla}\times\vec{E} = 0.$$

Again $\phi(\vec{r})$ is the energy/C gained by moving a charge into a fixed electrostatic field. Of course, when we move this charge into the system, we change the fields, and this change affects the next charge we move in.

Let's try to account this systematically. Consider an array of charges:



Q_i at $\vec{r}_i$

How much energy (work) did it take to assemble this distribution?

Analyze this in two ways:

First, put charge Q_1 at $\vec{r}_1$. This takes no work.

Next put charge Q_2 at $\vec{r}_2$. This takes

$$\Delta W = Q_2 \cdot \frac{Q_1}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|}$$

Next, put charge Q_3 at $\vec{r}_3$. This takes

$$\Delta W = Q_3 \left\{ \frac{Q_1}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_2|} \right\}$$

Now it is clear that the total work needed is

[It doesn't matter in what order we move the charges.]

$$W = \sum_{\text{pairs }(ij)} \frac{Q_i Q_j}{4\pi\epsilon_0 |\vec{r}_i - \vec{r}_j|}$$

this is conveniently written by replacing the sum over pairs (ij) by a sum over i and j. Each term in the sum over pairs appears twice in the independent sum over i and j. So compensate this by dividing by 2:

$$W = \frac{1}{2} \sum_{\substack{i,j \\ i\neq j}} \frac{Q_i Q_j}{4\pi\epsilon_0 |\vec{r}_i - \vec{r}_j|}$$

For a continuous charge distribution $\sum_i Q_i \to \int d^3x \, \rho(x)$

$$W = \frac{1}{2} \int d^3x_1 \, d^3x_2 \, \frac{1}{4\pi\epsilon_0} \frac{\rho(x_1)\,\rho(x_2)}{|\vec{X}_1 - \vec{X}_2|}$$

A region of size δ around $x_1 = x_2$ contributes:

$$\frac{[\rho(x_1)]^2}{4\pi\epsilon_0} \frac{\delta^3}{\delta} \to 0 \quad \text{as} \quad \delta \to 0$$

so we can drop the restriction $i \neq j$ in this case.

Here is another derivation of the formula for the energy needed to construct a charge distribution. Let α be a parameter: $0 < \alpha < 1$. α will characterize a stage in the process of building up the charge distribution. $\alpha = 0$ means we haven't started, $\alpha = 1$ means we are done. Build up the charge distribution uniformly so that at stage α the charge distribution is

$$\alpha \rho(\vec{x})$$

Let the work done up to this stage be $W(\alpha)$. The electrostatic potential at this stage is

$$\phi(\vec{r}) = \int d^3x_2 \; \frac{\alpha \rho(\vec{x}_2)}{4\pi\epsilon_0 |\vec{r} - \vec{x}_2|}$$

Now move in the next bit of charge: $d\rho(\vec{x}) = d\alpha \, \rho(\vec{x})$

this requires work:

$$dW = \int d^3x_1 \; d\alpha \, \rho(x_1) \cdot \phi(x_1)$$

$$= d\alpha \int d^3x_1 \, \rho(x_1) \int d^3x_2 \; \frac{\alpha \rho(x_2)}{4\pi\epsilon_0 |x_1 - x_2|}$$

Thus,

$$\frac{dW}{d\alpha} = \alpha \cdot \int d^3x_1\, d^3x_2 \; \frac{\rho(x_1)\,\rho(x_2)}{4\pi\epsilon_0 |\vec{x_1}-\vec{x_2}|}$$

the total work to be done is

$$W(1) = \int_0^1 d\alpha \; \frac{dW}{d\alpha} = \underbrace{\int_0^1 d\alpha \; \alpha}_{\frac{1}{2}} \int d^3x_1\, d^3x_2 \; \frac{\rho(x_1)\,\rho(x_2)}{4\pi\epsilon_0 |\vec{x_1}-\vec{x_2}|}$$

So

$$W = \frac{1}{2} \int d^3x_1\, d^3x_2 \; \frac{\rho(x_1)\,\rho(x_2)}{4\pi\epsilon_0 |\vec{x_1}-\vec{x_2}|}$$

in agreement with our other argument.

There is an amazing way to rearrange this formula. Imagine that we are talking about a localized distribution of charge, so that $\vec{E} \to 0 \quad \Phi \to 0$ as $r \to \infty$. Then, write W as

$$W = \frac{1}{2} \int d^3x \; \rho(x)\, \Phi(x)$$

Next, use $\quad \frac{\rho(x)}{\epsilon_0} = \vec{\nabla}\cdot\vec{E}$

$$W = \frac{\epsilon_0}{2} \int d^3x \; (\vec{\nabla}\cdot\vec{E})\, \Phi$$

now, by the product rule for differentiation

$$\vec{\nabla}(\vec{E}\phi) = \vec{\nabla}\cdot\vec{E}\,\phi + \vec{E}\cdot\vec{\nabla}\phi$$

then

$$W = \frac{\epsilon_0}{2}\int d^3x\ \vec{\nabla}(\vec{E}\phi) - \frac{\epsilon_0}{2}\int d^3x\ \vec{E}\cdot\vec{\nabla}\phi$$

the first term integrates to

$$\frac{\epsilon_0}{2}\int_S d^2x\ \hat{n}\cdot\vec{E}\phi$$ where S is a large sphere at radius R as $R\to\infty$ this $\sim R^2\cdot\frac{1}{R^2}\cdot\frac{1}{R}\to 0$

this leaves

$$W = \left(\text{energy of configuration of charges}\right) = \frac{\epsilon_0}{2}\int d^3x\ E^2$$

In our earlier formulae, the energy was contained in the interactions of charges. In this formula, we see that there is an equivalent picture in which the energy resides in the field and is delocalized throughout space.