Field Lines and Gauss' Law - 2

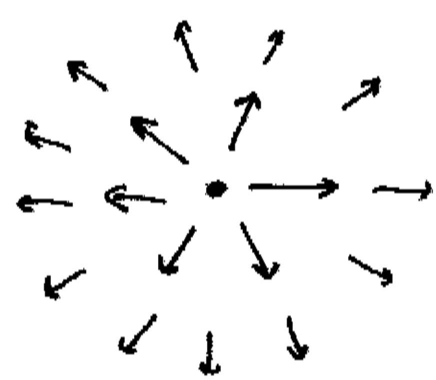
Now I would like to apply $\vec{\nabla}$ to the electric field determined by Coulomb's law: We saw that the electric field due to fixed charges is:

$$\vec{E}(\vec{r}) = \sum_i \frac{1}{4\pi\epsilon_0} \frac{Q_i}{|\vec{r}-\vec{r}_i|^2} \widehat{r-r_i}$$

By linearity, we can consider the elements of this sum one at a time. So begin with the $\vec{E}$ field of a point charge at $\vec{r}=0$:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

This looks like an outward flow from $\vec{r}=0$:



More explicitly:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} Q \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)$$

with $r^3 = [x^2+y^2+z^2]^{3/2}$

$$\frac{\partial E^x}{\partial x} = \frac{1}{r^3} - \frac{3}{2}\cdot 2\, \frac{x\cdot x}{r^5} = \frac{r^2-3x^2}{r^5} = \frac{y^2+z^2-2x^2}{r^5}$$

(away from the singularity at $r=0$)

Then $$\frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = \frac{(y^2+z^2-2x^2)+(x^2+z^2-2y^2)+(x^2+y^2-2z^2)}{r^5}$$

$$= 0$$

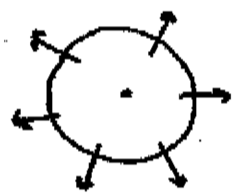
so

$\vec{\nabla}\cdot\vec{E}(r) = 0$ for $\vec{r} \neq 0$. Similarly, in the general case

$$\vec{\nabla}\cdot\vec{E}(r) = 0 \quad \text{at places where there is \underline{no} \underline{charge}}$$

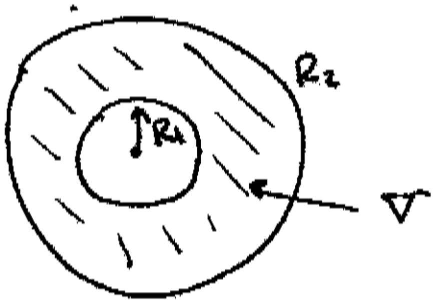
__But__, $\vec{E}$ field is definitely flowing away from $\vec{r}=0$. If we compute the flux of $\vec{E}$ through a large sphere of radius R

$$\Phi = \int d^2x\; \hat{n}\cdot\vec{E} = 4\pi R^2 \cdot \frac{Q}{4\pi\epsilon_0 R^2}$$



$$\Phi = \frac{Q}{\epsilon_0} \quad \text{independently of } R.$$

we know that Φ must be independent of R, since the net flux into the region between two spheres



$$\Phi = \Phi_{R_1} - \Phi_{R_2} = \int d^2x\; \hat{n}\cdot\vec{E} = \int d^3x\; \vec{\nabla}\cdot\vec{E} = 0$$

Now, if the flux out of a sphere of radius R is $Q/\epsilon_0 > 0$ and $\vec{\nabla}\cdot\vec{E} = 0$ everywhere but at $\vec{r} = 0$, we had better have

$$\vec{\nabla}\cdot\vec{E}\,(\vec{r}=0) = \text{ENORMOUS} \quad \text{s.t.} \quad \int d^3x\, \vec{\nabla}\cdot\vec{E} = \frac{Q}{\epsilon_0}$$

This behavior will recur many times in this course, and it is useful to formalize it. Consider first the analogous behavior in 1 dimension. We can consider a function $\delta(x)$ — more properly, a limit of continuous functions — with the properties:

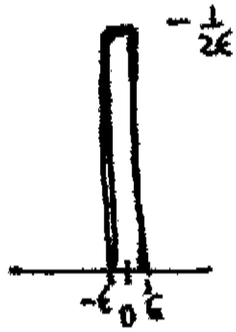
the Dirac <u>δ-function</u>

$$\delta(x) = 0 \qquad \text{if} \quad x \neq 0$$

$$\int_{-a}^{b} dx\, \delta(x) = 1 \qquad a, b > 0$$

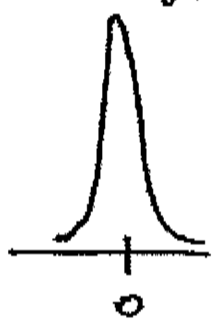
One way to look at $\delta(x)$ is as the limit

$$\delta(x) = \lim_{\epsilon \to 0} \begin{cases} 0 & |x| > \epsilon \\ \frac{1}{2\epsilon} & |x| < \epsilon \end{cases}$$



Other useful definitions are

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\,\epsilon} e^{-x^2/2\epsilon^2}$$



$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$$

Typically, though, it will suffice to think about $\delta(x)$ just as a huge peak at $x = 0$. Two useful properties of $\delta(x)$ are:

$$\int_{-a}^{b} dx\ f(x)\ \delta(x) = f(0) \qquad \text{for any continuous function } f(x)$$

$$\delta(g(x)) = \sum_{0} \frac{1}{|g'(x_0)|} \delta(x - x_0) \qquad \text{where } x_0 \text{ are the zeros of } g(x)$$

For the proof of the second statement, let $y = g(x)$. Consider $\int dy\ \delta(y) = 1$ for an interval that contains one of the x_0. Now $dy = g'(x)\, dx$. Assume first $g'(x_0) > 0$

$$1 = \int dy\ \delta(y) = \int dx\ g'(x)\ \delta(g(x)) = g'(x_0) \int dx\ \delta(g(x))$$

so

$$\int dx\ \delta(g(x)) = \frac{1}{g'(x_0)} \qquad \text{as required}$$

if $g'(x_0) < 0$

$$\int_{-\eta}^{+\eta} dy\ \delta(y) = g'(x_0) \int_{x_0 + \eta |g'(x_0)|}^{x_0 - \eta |g'(x_0)|} dx\ \ \delta(g(x))$$

since we are integrating over x in the wrong direction, we get an extra (-1).

In 3-dimensions, we can define

$$\delta^{(3)}(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

This satisfies

$$\int_v d^3x\; \delta^{(3)}(\vec{r}) = 1 \qquad\qquad \int_v d^3x\; f(\vec{r})\,\delta^{(3)}(\vec{r}) = f(\vec{0})$$

if v contains $\vec{r}=0$

This is the object we need for $\vec{\nabla}\cdot\vec{E}$. For a point charge at $\vec{r}=0$

$$\vec{\nabla}\cdot\vec{E} = \frac{Q}{\epsilon_0}\,\delta^{(3)}(\vec{r})$$

For several point charges

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\sum_i Q_i\,\delta^{(3)}(\vec{r}-\vec{r}_i)$$

We can now pass to the limit of a continuous distribution of charge

$\rho(\vec{x})$ = density of electric charge (C/m^3)

$$\sum_i Q_i \longrightarrow \int d^3x\; \rho(\vec{x})$$

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{\epsilon_0}\int d^3x\; \rho(\vec{x})\,\delta^{(3)}(\vec{r}-\vec{x}) = \frac{1}{\epsilon_0}\,\rho(\vec{r})$$

This equation summarizes our analysis ("Gauss' law")

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \quad , \text{ in empty space } \quad \vec{\nabla}\cdot\vec{E} = 0$$

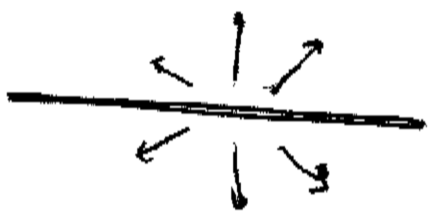
We can think of point charges as limits as size $\to 0$ of continuous charge distributions; that makes the formulae with δ-functions special cases of the above.

The integral form of Gauss' law is

$$\int_V d^3x \, \underbrace{\vec{\nabla}\cdot\vec{E}}_{\substack{\text{flux of } \vec{E} \text{ through} \\ S = \partial V}} = \frac{1}{\epsilon_0} \int_V \underbrace{d^3x \, \rho}_{\text{total charge in } V}$$

$$\Phi = \frac{Q}{\epsilon_0}$$

By cleverly choosing surfaces and volumes and using symmetry, we can use this equation to recover the expressions for the $\vec{E}$ produced by charge distributions that we found in the first lecture. Consider first the infinite wire with a charge density ρ C/m. By symmetry, it is

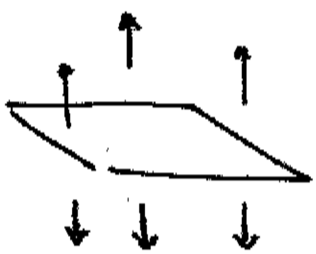


"obvious" that the $\vec{E}$ field is radial, $\perp$ to the wire.
Let the magnitude of the $\vec{E}$ field at distance $r_\perp$ be $E(r_\perp)$
then the flux through a cylinder of radius $r_\perp$:

is
$$\Phi = 2\pi r_\perp l \cdot E(r_\perp) = Q_{\text{enclosed}}/\epsilon_0 = \rho l/\epsilon_0$$

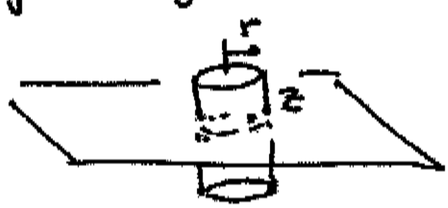
so indeed
$$E(r_\perp) = \frac{\rho}{2\pi\epsilon_0 r_\perp} \quad ✓$$

Next, consider a plane with a uniform charge density ρ C/m². In this case it is "obvious" that the



E field is $\perp$ to the plane and depends only on the height z.

Then the flux through a cylinder that goes out to z:



$$\pi r^2 E(z) \cdot \underbrace{2}_{\text{top+bottom}} = \underbrace{Q/\epsilon_0}_{\text{enclosed}} = \pi r^2 \rho/\epsilon_0$$

$$E(z) = \frac{\rho}{2\epsilon_0} \quad \text{as before.}$$

This calculations are elegant, but they are also somewhat artificial. Not only are the systems especially simple, but we have used symmetry arguments to fill in for some ignorance. Gauss' law only tells us about the component of $\vec{E}$ normal to the surface. We need another equation to give us the component of $\vec{E}$ parallel to the surface. We'll discuss that next time.