Topology of the $\vec{A}$ field

In this final lecture, I will discuss some exotic aspects of magnetostatics and the vector potential $\vec{A}$. Up to now, you might have thought that $\vec{A}$ is a purely auxiliary quantity, and that everything we do with $\vec{A}$ we could just as well have done with $\vec{B}$. In this lecture, I will explain that $\vec{A}$ is essential. I will also explain why the fundamental character of $\vec{A}$ does not imply $\vec{\nabla}\cdot\vec{B} = 0$ everywhere.

In classical particle mechanics we can just use $\vec{B}$. A classical point particle obeys the equations of motion:

$$m\dot{\vec{v}} = q\vec{E} + q\vec{v}\times\vec{B}$$

To write the equation of a quantum mechanical particle, however, we must invoke $\vec{A}$. Obviously, I cannot teach you all of quantum mechanics here. But let me write down the rules — without proof — and then we will see how $\vec{A}$ fits in.

In quantum mechanics, a particle does not have a definite position. Rather, the "location" of a particle is a complex-valued function of $\vec{x}$ called the Schrödinger wavefunction $\Psi(\vec{x})$. The integral over a region R

$$P(R) = \int_R d^3x \, |\Psi(x)|^2$$

gives the probability of finding the particle in R. Of course

$$\int_{\text{all space}} d^3x \, |\psi(\vec{x})|^2 = 1 .$$

To find the wavefunctions $\psi(\vec{x})$ corresponding to stationary states, such as the energy levels of an atom, we solve a Sturm-Liouville problem for an operator $\underline{H}$ – the Hamiltonian operator – which represents the energy. $\underline{H}$ acts on the vector space of complex-valued functions with the inner product

$$\langle \psi, \chi \rangle = \int d^3x \; \psi^* \chi$$

For a wavefunction, total probability 1 is the same thing as unit length:

$$\langle \psi, \psi \rangle = 1$$

$$\hbar = \frac{h}{2\pi}$$

H is built from the operators $\underline{x}$ and $\underline{p}$:

$$\underline{x}\, \psi(x) = x\, \psi(x) \qquad \underline{p}\, \psi(x) = -i\hbar \frac{\partial}{\partial x} \psi(x)$$

For macroscopic motions $\underline{x}$ and $\underline{p}$ correspond to the classical x and p of the particle. But note that $\underline{x}$ and $\underline{p}$ are operators and do not commute:

$$\underline{x}\,\underline{p}\,\psi = x\,(-i\hbar)\frac{\partial}{\partial x}\psi \qquad \underline{p}\,\underline{x}\,\psi = (-i\hbar)\frac{\partial}{\partial x}(x\psi)$$

so

$$\underline{x}\,\underline{p} - \underline{p}\,\underline{x} = i\hbar$$

In general, we write $\underline{A}\,\underline{B} - \underline{B}\,\underline{A} \equiv [\underline{A}, \underline{B}]$.

What, exactly, should we write for $\underline{H}$. It turns out that the H which gives the correct classical limit for macroscopic motions is the one for which the prescription:

$$i\hbar \dot{\underline{x}} = [\underline{x}, \underline{H}] \qquad i\hbar \dot{\underline{p}} = [\underline{p}, \underline{H}]$$

gives the classical equations of motion. For example, the $\underline{H}$ for a free particle is:

$$\underline{H} = \frac{\underline{p}^2}{2m}$$

then

$$i\hbar \dot{x} = [\underline{x}, \tfrac{1}{2m}\underline{p}^2] = \frac{1}{2m} \cdot 2 i\hbar \underline{p} \quad \Rightarrow \quad \dot{x} = \frac{p}{m}$$

$$i\hbar \dot{p} = [\underline{p}, \frac{\underline{p}^2}{2m}] = 0 \quad \Rightarrow \quad \dot{p} = 0$$

and these are the equations of a free particle. H can be written more explicitly as

$$\underline{H} = -\frac{1}{2m}\hbar^2 \frac{\partial^2}{\partial x^2}$$

The eigenfunctions of this operator are of the form e^{ikx},

$$\underline{H}\, e^{ikx} = \frac{\hbar^2 k^2}{2m} e^{ikx}$$

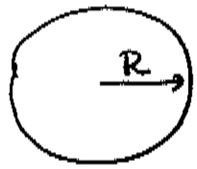
$$\underline{p}\, e^{ikx} = \hbar k\, e^{ikx}$$

The oscillation e^{ikx} has wavelength $k\lambda = 2\pi \qquad k = 2\pi/\lambda$

so $\qquad p = h/\lambda \quad$ (de Broglie's relation)

and $\qquad$ Energy $= p^2/2m = \frac{1}{2}mv^2$ as required

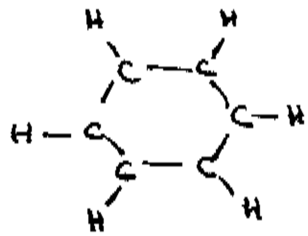
For a particle on a ring:



$$\psi = e^{ikR\phi} \quad \text{must be periodic} \quad = e^{in\phi}$$

then
$$k = \frac{n}{R} \qquad p = \frac{n\hbar}{R} \qquad E = \frac{\hbar^2}{2mR^2} n^2$$

Energy levels quantized in this way are found, for example, in the energy level spectrum of the benzene molecule:



For an electrostatic field, it makes sense to write

$$H = \frac{p^2}{2m} + q\phi(\underline{x})$$

where $\phi(x)$ is the electrostatic potential. As before

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{p}{m}$$

But now
$$\dot{p} = \frac{1}{i\hbar}[p, H] = \frac{1}{i\hbar}\left(-i\hbar \frac{\partial}{\partial x}\right)(q\phi) = q\left(-\frac{\partial}{\partial x}\phi\right)$$

so
$$\dot{p} = qE \quad \text{as required}$$

This is fine for $\vec{E}$; what about $\vec{B}$? Now we have to work in 3 dimensions and use indices.

$$\underline{x}^j \psi = x^j \psi \qquad \underline{p}^j \psi = -i\hbar \frac{\partial}{\partial x^j} \psi$$

I claim that an appropriate H which includes a magnetostatic field is:

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2$$

Check:

$$\dot{\underline{x}}^j = \frac{1}{i\hbar} [\underline{x}, H] = \frac{1}{m} (\vec{p} - q\vec{A})^j$$

$$\dot{\underline{p}}^j = \frac{1}{i\hbar} [\underline{p}, H] = \frac{1}{m} (\vec{p} - e\vec{A})^k \left(-\frac{\partial}{\partial x^j}\right)(-qA^k)$$

$$= q v^k \left(\frac{\partial}{\partial x^j} A^k\right)$$

then

$$m\dot{v}^j = \dot{\vec{p}}^j - q\dot{\vec{A}}^j = q v^k \frac{\partial}{\partial x^j} A^k - q v^k \frac{\partial}{\partial x^k} A^j$$

$$= q [\vec{v} \times (\vec{\nabla} \times \vec{A})]^j$$

so we do recover the Lorentz force equation using the rules of quantum mechanics with this $\underline{H}$.

We can write the Hamiltonian operator more explicitly as

$$H = -\frac{\hbar^2}{2m} \left(\vec{\nabla} - i\frac{q}{\hbar}\vec{A}\right)^2$$

This Hamiltonian satisfies an amazing symmetry condition. Let $\underline{H}$ act on a Schrödinger wavefunction $\psi(x)$.

Now consider the action of $H[A]$ on

$$\psi'(x) = e^{i\alpha(x)}\psi(x)$$

where we rotate the phase of ψ independently at each point in space.

$$H[A]\psi' = -\frac{\hbar^2}{2m}\left(\vec\nabla - i\frac{q}{\hbar}\vec A\right)^2 e^{i\alpha(x)}\psi(x)$$

$$= e^{i\alpha(x)}\left(-\frac{\hbar^2}{2m}\left(\vec\nabla + i(\vec\nabla\alpha) - i\frac{q}{\hbar}\vec A\right)^2\psi(x)\right)$$

There is an extra term, but we can compensate it by making a gauge transformation of A

$$\vec A' = \vec A + \frac{\hbar}{q}\vec\nabla\alpha$$

Then

$$H[A']\psi' = e^{i\alpha(x)}H[A]\psi$$

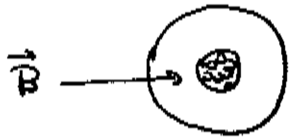
and if ψ is an eigenvector of $H[A]$: $\quad H[A]\psi(x) = E\psi(x)$

then ψ' is an eigenvector of $H[A']$ with the same eigenvalue:

$$H[A']\psi'(x) = E\psi'(x)$$

This is the sense in which the predictions of quantum mechanics depend only on the magnetic field, and not on the particular choice of $\vec A$.

However, as I will now show, it is possible for a magnetic field to change observable predictions of quantum mechanics even though the quantum particle moves in a region where $\vec B = 0$. Consider the example of a particle on a ring (as discussed above) and put a small solenoid through the ring, carrying flux $\Phi = \int d^2x\, \hat{n}\cdot\vec{B}$



Our quantum particle will be confined to the circle, where $\vec{B} = 0$. Nevertheless, there is an $\vec{A}$ on the ring, since

$$\oint d\vec{l}\cdot\vec{A} = \Phi \quad \Rightarrow \quad \vec{A} = \frac{\Phi}{2\pi R}\hat{\phi} \quad \text{up to a gauge transformation}$$

It is not hard to solve the eigenvalue problem for H in this situation:

$$H = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - i\frac{q}{\hbar}\vec{A}\right)^2\psi(\phi)$$

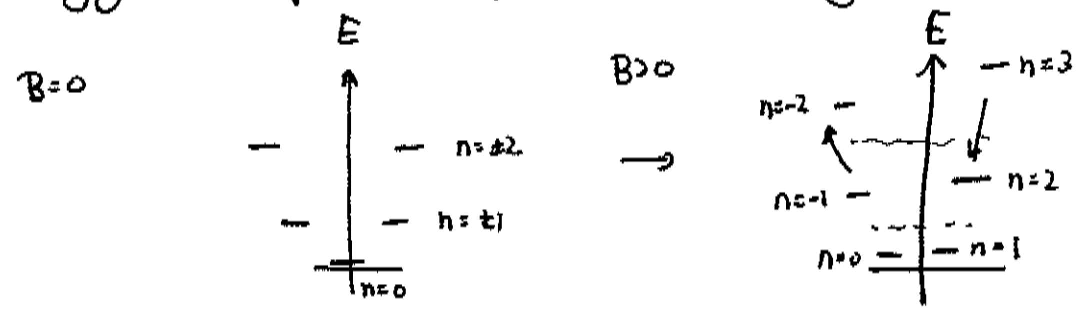
$\hat{\phi}$ component of $(\vec{\nabla} - \frac{q}{\hbar}\vec{A})$

$$= -\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{\partial}{\partial\phi} - i\frac{q}{\hbar}\frac{\Phi}{2\pi R}\right)^2\psi$$

so $\psi = e^{in\phi}$ is an eigenfunction obeying the periodic boundary condition, and the corresponding energy eigenvalue is

$$E = \frac{\hbar^2}{2mR^2}\cdot\left(n - \frac{q}{2\pi\hbar}\Phi\right)^2$$

The energy level spectrum shifts with increasing Φ or B

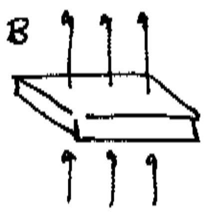


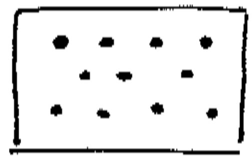
This shift of energy levels is observable. This is, in fact, the principle of operation of the SQUID magnetometer. Note that there is no effect on the spectrum when Φ is a multiple of the "London flux quantum"

$$\Phi_L = \int d^2x \, \hat{n} \cdot B = \frac{2\pi\hbar}{q}$$

As Fairbanks proved experimentally at Stanford, a superconductor can admit magnetic field which penetrates the superconductor in narrow "flux tubes", each of which carries one London quantum of flux

$$\frac{2\pi\hbar}{q} \quad \text{with} \quad q = 2e \quad (\text{from } e^- \text{ pairing!})$$





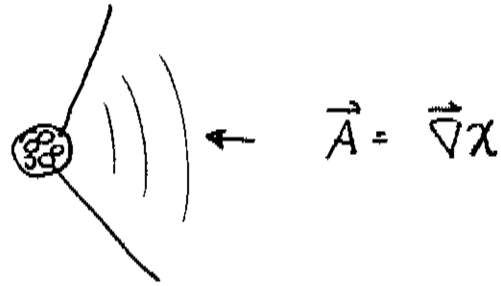
B Φ

How is this possible? If $\vec{B} = 0$, can't we remove $\vec{A}$ by a gauge transformation. It is true that $\vec{\nabla}\times\vec{A} = 0$ outside the solenoid, but this is a case where $\vec{\nabla}\times\vec{A} = 0$ does not imply $\vec{A} = \vec{\nabla}\chi$, due to the fact that the region we consider is not simply connected. In fact, in any small region of ϕ

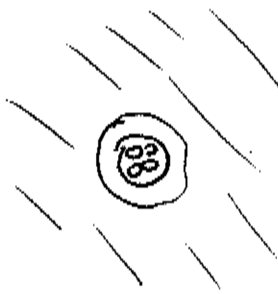
$$\vec{A} = \frac{\Phi}{2\pi R}\hat{\phi} \quad \Rightarrow \quad \vec{A} = \vec{\nabla}\chi \text{ with}$$

$$\chi = \frac{\Phi}{2\pi}\phi \qquad \left(\vec{\nabla} = \frac{\hat{\phi}}{R}\frac{\partial}{\partial\phi} + \cdots \right)$$

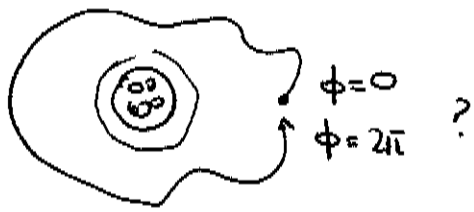
Again, this works in any angular wedge coming out of the solenoid



But ϕ is not a function defined on the whole region we are considering



because it is not single valued: Moving continuously from a point where $\phi = 0$, we can go around the solenoid and find $\phi = 2\pi$



Another way to see this is that the integral

$$\Phi = \oint d\vec{l} \cdot \vec{A}$$

is unchanged by a gauge transformation $A' \to \vec{A} + \frac{\hbar}{q}\vec{\nabla}\alpha$

$$\Delta\Phi = \oint d\vec{l} \cdot \frac{\hbar}{q}\vec{\nabla}\alpha = \frac{\hbar}{q}\left[\alpha(2\pi) - \alpha(0)\right] = 0$$

so physical predictions can depend on Φ even though Φ

cannot be represented locally in terms of $\vec{B}$.

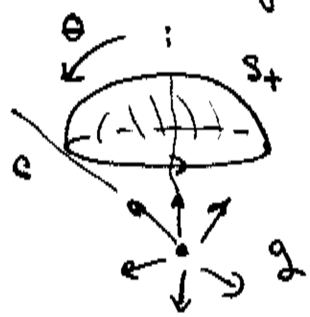
Now we have the preparation to discuss a very interesting question: If we must regard $\vec{A}$ as fundamental, does this imply that $\vec{\nabla}\cdot\vec{B}=0$ everywhere? To analyze this question, I would like to construct an $\vec{A}$ field whose curl is the $\vec{B}$ field of a <u>magnetic</u> <u>monopole</u>:

$$\vec{B} = \frac{\mu_0}{4\pi} g \cdot \frac{\hat{r}}{r^2} \qquad g = \text{magnetic charge}.$$

This is obviously impossible, because $\vec{\nabla}\cdot\vec{B}\neq 0$. However, we can find an $\vec{A}$ in some limited regions. Consider, for example

$$\vec{A}_+ = \frac{\mu_0 g}{4\pi} \frac{1-\cos\Theta}{r\sin\Theta}\hat{\phi}$$

This field configuration is set up so that the integral $\oint d\vec{l}\cdot\vec{A}$ around a loop equals the flux of the magnetic monopole field through the spherical cap covering the loop from above:



$$\oint_C d\vec{l}\cdot\vec{A} = 2\pi r\sin\Theta\, A_{+\phi} = \int_{S_+} d^2x\, \hat{n}\cdot B$$

$$= \int_0^\Theta 2\pi\, d\Theta' \sin\Theta' r^2 \frac{\mu_0}{4\pi} g \frac{1}{r^2}$$

$$= 2\pi\,(1-\cos\Theta)\frac{\mu_0}{4\pi} g$$

by acting on $\vec{A}_+$ with $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$

you can also show explicitly that

$$\vec{\nabla}\times\vec{A}_+ = \frac{\mu_0}{4\pi} g \frac{\hat{r}}{r^2} = \vec{B}$$

Notice ~~that~~ A_+ is regular for $\cos\theta > -1$; in particular, since $1-\cos\theta \sim \theta^2/2$, $\sin\theta \sim \theta$, $\vec{A}_+ \to 0$ as $\theta \to 0$ where $\hat{\phi}$ is not defined. However, $\vec{A}_+$ is singular on the line $\theta = \pi$ where $\frac{1-\cos\theta}{\sin\theta} \to \infty$.

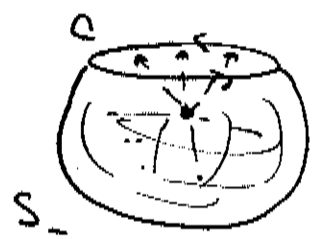
Another interesting choice for $\vec{A}$ is

$$\vec{A}_- = -\frac{\mu_0 g}{4\pi}\frac{1+\cos\theta}{r\sin\theta}\hat{\phi}$$

This field satisfies

$$\oint_C d\vec{l}\cdot\vec{A}_- = \int_{S_-} d^2x\ \hat{n}\cdot\vec{B}$$

where S_- is the spherical cap covering the loop from underneath



and $\hat{n}$ points inward to give the correct orientation on C:

$$\oint_C d\vec{l}\cdot\vec{A}_- = 2\pi r\sin\theta\, A_{-\phi} = \int_{S_-} d^2x\, \hat{n}\cdot B$$

$$= -\ 2\pi(1+\cos\theta)\frac{\mu_0 g}{4\pi}$$

A_- is regular in the region $\theta > 0$ (and in particular, at $\theta = \pi$) but is singular as $\theta \to 0$. In any small region away from $\theta = 0, \pi$, $\vec{A}_+$ and $\vec{A}_-$ are related by a gauge transformation

$$\vec{A}_+ - \vec{A}_- = \frac{\mu_0 g}{2\pi r\sin\theta}\hat{\phi} = \vec{\nabla}\left(\frac{\mu_0 g}{2\pi}\phi\right)$$

However, this gauge transformation cannot be performed globally even if we exclude regions around $\theta = 0$, $\theta = \pi$, because ϕ is not defined globally.

The situation that we meet here is analogous to the problem of defining a coordinate system on a sphere. In any small region of the sphere, there is a way to put down a well-defined set of coordinates. However, any such coordinate system eventually becomes singular or multiple-valued when it is extended far enough. Ever since Columbus, this has been considered not to be a problem. We cover the sphere with different coordinate systems, one for each of several overlapping regions:

As long as there is a well-defined way to change coordinates in the regions of overlap, the system is consistent.

The situation of having a set of mutually consistent local coordinate systems gives the mathematical definition of a manifold. If there is no globally defined coordinate system, we say that the manifold has nontrivial topology. Typically, this is associated with the presence in the manifold of a loop, sphere, etc. that cannot be contracted to a point.

The magnetic monopole field, written in terms of $\vec{A}$, is also viewed by mathematicians as a space of nontrivial topology. It corresponds to a generalization of a manifold called a fiber bundle. In this structure $\vec{\nabla}\cdot\vec{B} = 0$ implies that, locally, we can find an $\vec{A}$ field such that $\vec{\nabla}\times\vec{A} = \vec{B}$. However, such an $\vec{A}$ does not exist globally in the region $r > 0$ away from the center of the monopole. But still we can find $\vec{A}$ in each of several regions and relate these $\vec{A}$ choices by gauge transformations.

Now we come to an interesting question: A classical point particle, which feels only $\vec{B}$, can clearly be defined consistently in a magnetic monopole field. But it is not clear that we can describe a quantum mechanical particle in the field of a magnetic monopole.

Let's try. First, write the Hamiltonian operator $H[\vec{A}_+]$. Find an eigenfunction $\psi_+(x)$:

$$H[\vec{A}_+]\,\psi_+(x) = E\,\psi_+(x)$$

$\psi_+(x)$ must be regular for $\theta < \pi$, but it is not clear what boundary condition ψ_+ should satisfy as $\theta \to \pi$.

Now write the Hamiltonian $H[\vec{A}_-]$. Find an eigenfunction $\psi_-(x)$ with the same energy

$$H[\vec{A}_-]\,\psi_-(x) = E\,\psi_-(x)$$

$\psi_-(x)$ must be regular for $\theta > 0$, but it is not clear what boundary condition ψ_- should satisfy as $\theta \to 0$.

To fix boundary conditions at $\theta \to 0$ and $\theta \to \pi$, we have to insist that the two wavefunctions we have found are two different views of the same solution. For this to be true, ψ_+ and ψ_- must be equal up to the gauge transformation

$$\psi_+(x) = e^{i\alpha(x)}\,\psi_-(x) \qquad 0 < \theta < \pi$$

where

$$\frac{\hbar}{q}\vec{\nabla}\alpha = \vec{A}_+ - \vec{A}_-$$

But, we computed α and found $\alpha = \frac{\mu_0 g q}{2\pi\hbar}\phi$. So, if ψ_+ and ψ_- are representations of the same wavefunction

$$\psi_+(x) = e^{i\left[\frac{\mu_0 g q}{2\pi\hbar}\right]\phi}\,\psi_-(x)$$

But, if the coefficient of ϕ in the exponent is a general real number, either ψ_+ or ψ_- cannot be single-valued as a function of Φ. So, finally, we can consistently describe a quantum mechanical particle of charge q in the field of a magnetic monopole of charge g only if

$$\frac{\mu_0 g q}{2\pi\hbar} = \text{integer}$$

Dirac found this result in 1931: If the equations of Nature allow the production of a magnetic monopole of charge g, then all electric charges in the universe are quantized according to

$$q = \frac{2\pi\hbar}{\mu_0 g} \cdot n$$

Similarly, any possible magnetic monopole in Nature has a charge which obeys

$$g = \frac{2\pi\hbar}{\mu_0 e} \cdot m$$

This is an amazing idea, and it is worth asking how it stands up today.

First, extensive searches for magnetic monopoles have turned up nothing. Blas Cabrera has set a limit on magnetic monopoles in the cosmic ray flux

$$\text{flux} < 5\times10^{-14} \,/\text{cm}^2\,\text{sec sr}$$

a stronger limit, the Parker bound, is based on the idea that a nonzero density of magnetic charges in the galaxy would neutralize the galactic magnetic fields. This leads to a limit

$$\text{monopole flux} \lesssim 10^{-18}/\text{cm}^2\,\text{sec}\,\text{sr}$$

(depending on the monopole velocity). To estimate the limits on the density, multiply by a velocity of $1.2\times10^{-3}\,c$, our velocity with respect to the cosmic microwave background, and by 4π. Then we get

$$\rho_M \lesssim 10^{-11}/\text{cm}^3$$

to be compared to $\rho_{\text{proton}} \cong \rho_{\text{electron}} \cong 1/\text{cm}^3$ in deep space.

On the theoretical side, quantization of electric charge is now understood in another way. If the phase rotation symmetry

$$\psi(x) \rightarrow e^{i\alpha(x)}\,\psi(x)$$

is embedded in a non-commutative group of continuous symmetries, charge quantization can be derived from quantum mechanics in the same way that angular momentum quantization is derived. Then we do not need magnetic monopoles to explain the fact that

$$\text{electron charge} / \text{proton charge} = 1 \text{ to accuracy } 10^{-21}\,!$$

The generalization of Maxwell's equations that accounts for this is called a "non-Abelian gauge theory" or Yang-Mills theory.

However, magnetic monopoles still have a place in the story. 't Hooft and Polyakov studied non-Abelian gauge theories that contain electromagnetism and showed that these theories have classical field configurations with magnetic charge. In fact, magnetic charge is quantized in units of

$$g = \frac{2\pi\hbar}{\mu_0} \frac{1}{e_m}$$

where e_m is the minimal value of electric charge!

Today, most particle theorist would give you the following conventional wisdom about magnetic monopoles: Electromagnetism is unified with other fundamental interactions at the energy scale of "grand unification": $M_G = 2 \times 10^{16}$ GeV (0.93 GeV = $m_{proton} c^2$). The 't Hooft-Polyakov magnetic monopole in this model has

$$g = 2\pi\hbar / \mu_0 e$$

and $mc^2 \cong 100 \cdot m_G \cong 10^{18}$ GeV. Thus, there can be magnetic monopoles in Nature, but they are very heavy. Magnetic monopoles were produced in pairs in the Big Bang, but their density was diluted by inflation and — since the reheating temperature at the end of the inflationary period of cosmology was at most about 10^{12} GeV, new monopoles were not made after inflation. Thus, there should not be any in the cosmic rays. Of course, many aspects of this story are extremely speculative. Maybe you can put it on a better footing — or refute it!