Field Lines and Gauss' Law

Sept 29.

We can get some more insight into the $\vec{E}$ field by thinking of it as a flow, emerging from positive electric charges and pouring into negative electric charges. To quantify this, we need to review some mathematical concepts.

A basic mathematical object is $\vec{\nabla}$, the directional derivative

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Applying $\vec{\nabla}$ to a scalar quantity t, we obtain

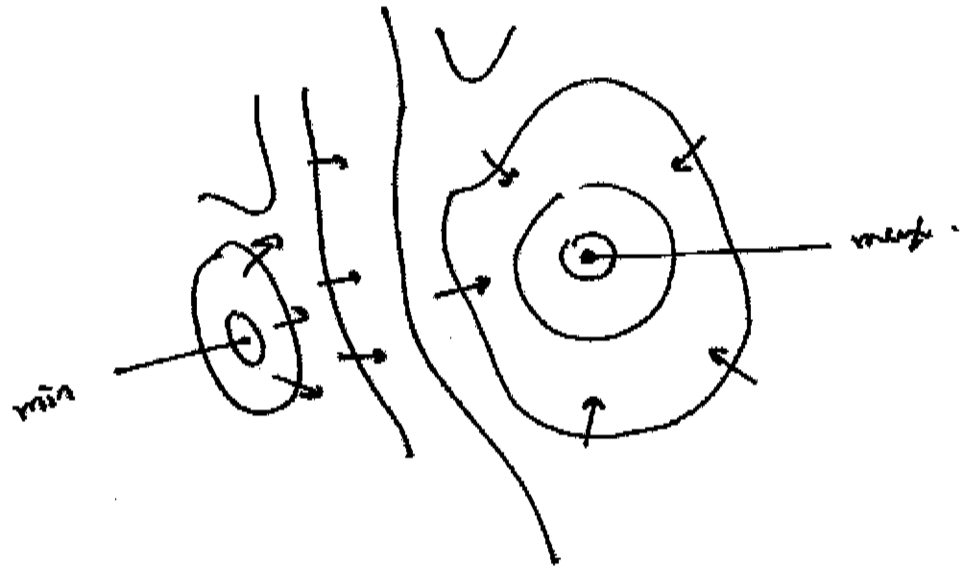
$$\vec{\nabla} t = \hat{x}\frac{\partial t}{\partial x} + \hat{y}\frac{\partial t}{\partial y} + \hat{z}\frac{\partial t}{\partial z} = \overrightarrow{\text{grad}}\; t$$

the gradient of $t(x,y,z)$. $\vec{\nabla} t(\vec{x})$ is a vector that points in the direction of the steepest change in $t(\vec{x})$. If $\vec{v}\cdot\vec{\nabla} t = 0$, t is constant along the direction of $\vec{v}$. If $\vec{\nabla} t = 0$

$$\frac{\partial t}{\partial x} = 0 = \frac{\partial t}{\partial y} = \frac{\partial t}{\partial z}$$

so we must be at a minimum, a maximum, or a saddle point of t. For example, here are some vectors $\vec{\nabla} t$

on a contour map of t



Apply $\vec{\nabla}$ to a vector $\vec{\omega}(x)$, we obtain an object with two vector indices. Particularly interesting combinations are

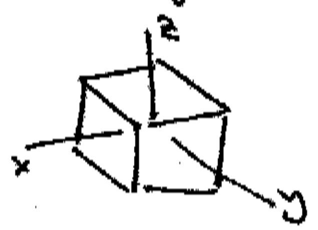
$$\vec{\nabla}\cdot\vec{\omega} = \text{div}\ \omega = \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z}$$

Representing the vectors with indices $i,j = xyz$ or 123 and assuming a sum over repeated indices (Einstein's convention)

$$\vec{\nabla}\cdot\vec{\omega} = \sum_{i=1}^{3} \frac{\partial}{\partial x^i}\omega_i = \frac{\partial}{\partial x^i}\omega_i$$

or just.

The divergence can be interpreted in terms of fluid flows.
Consider a small cube of side a aligned with the x, y, z axes.



Think of $\vec{\omega}$ in terms of the flow of a fluid. Let

$$\vec{\omega}(x) = \rho(x)\vec{v}(x) \qquad (\mathrm{kg}/\mathrm{m}^3)\cdot(\mathrm{m}/\mathrm{sec})$$

$$= \mathrm{kg}/\mathrm{m}^2/\mathrm{sec}$$

crossing a small surface in the flow

Usually, we call this $\vec{j}$ the <u>current</u>.

Given a surface S, we define the <u>flux</u> through S as

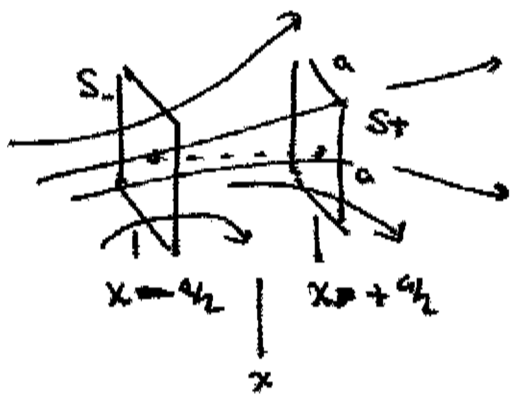
$$\Phi = \int_S d^2x\, \hat{n}\cdot\vec{j}$$

$\hat{n}$ is a unit vector normal to the surface.

$$\Phi = \int d^2x\, \hat{n}\cdot\vec{v}\rho = \mathrm{kg}/\mathrm{m}^3 \cdot \mathrm{m}^2 \cdot \mathrm{m}/\mathrm{sec} = \mathrm{kg}/\mathrm{sec}$$

is the rate at which matter is flowing through S.

Now consider a small cube and compute the net flow of matter into the cube: For the two surfaces $\perp$ to $\hat{x}$



Net flux in $= \int_{S_+} d^2x\,(\hat{-x})\cdot\vec{j}(x+a/2) + \int_{S_-} d^2x\,(\hat{x})\cdot\vec{j}(x-a/2)$

$a \to 0 \qquad \cong a^2\left[-j_x(x+a/2) + j_x(x-a/2)\right]$

$\cong a^3\left[-\frac{\partial}{\partial x} j_x\right]$

Including also the surfaces $\perp$ to $\hat{y}$ and $\hat{z}$:

Net matter transfer into the cube/sec $= a^3\left(-\frac{\partial}{\partial x}j_x - \frac{\partial}{\partial y}j_y - \frac{\partial}{\partial z}j_z\right)$

$= -a^3\,(\vec{\nabla}\cdot\vec{j})$

The amount of matter in the cube is

$$\int_{\text{cube}} d^3x\,\rho \cong a^3\rho$$

Then

$$\frac{d}{dt}(\text{matter in cube}) \cong a^3\frac{\partial\rho}{\partial t} \cong -a^3(\vec{\nabla}\cdot\vec{j})$$

sending $a \to 0$, these approximations become exact and we obtain "equation of continuity"

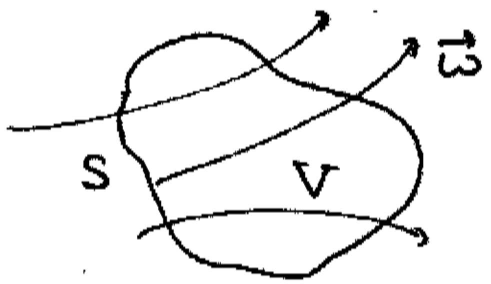
$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

This is a differential equation that expresses the overall conservation of matter in the fluid: It expresses the fact that any change in density requires that matter flow to supply the difference.

Stacking together a large number of small cubes, we obtain the integral formula, the divergence theorem

$$\Phi = \int_S d^2x\ \hat{n}\cdot\vec{\omega} = \int_V d^3x\ \vec{\nabla}\cdot\vec{\omega}$$

This theorem is actually valid for any volume V enclosed by a surface S, and for any (differentiable) vector field $\vec{\omega}$.



Using the divergence theorem, we may integrate the continuity equation over any region V:

$$\int_V d^3x\ \frac{\partial \rho}{\partial t} = -\int_V d^3x\ \vec{\nabla}\cdot\vec{j}$$

$$\underbrace{\frac{d}{dt}\left[\int_V d^3x\ \rho\right]}_{\text{total amt of matter in } V} = -\underbrace{\left[\int_S d^2x\ \hat{n}\cdot\vec{j}\ \right]}_{\text{flux of } \vec{j} \text{ out of } V \text{ through } S}$$

If $\frac{\partial \rho}{\partial t} = 0$ (ρ = const), then $\vec{\nabla}\cdot\vec{j} = 0$. A vector field $\vec{w}$ satisfying $\vec{\nabla}\cdot\vec{w} = 0$ is the flow field of an incompressible fluid.

A second useful action of $\vec{\nabla}$ on $\vec{w}$ is the curl

$$\vec{\nabla}\times\vec{w} = \left(\frac{\partial}{\partial y} w_z - \frac{\partial}{\partial z} w_y \,,\; \frac{\partial}{\partial z} w_x - \frac{\partial}{\partial x} w_z \,,\; \frac{\partial}{\partial x} w_y - \frac{\partial}{\partial y} w_x \right)$$

I will also write this as

$$(\vec{\nabla}\times\vec{w})^i = \epsilon^{ijk} \frac{\partial}{\partial x^j} w^k$$

where ϵ^{ijk} is the 3-index antisymmetric symbol.

$$\epsilon^{123} = +1 \qquad \epsilon^{213} = -1 \qquad \epsilon^{231} = +1$$

similarly, antisymmetric under any interchange of 2 indices.

$\epsilon^{113} = 0$ similarly $\epsilon = 0$ if two indices are equal.

If you write out $\epsilon^{ijk} \frac{\partial}{\partial x^j} w^k$, substituting all possible values for i,j,k, you will see that it reproduces the definition of the curl. You can also verify the following properties of ϵ, which we will use later in the course:

$$\epsilon^{ikl}\,\epsilon^{jkl} = 2\,\delta^{ij} \qquad \text{sum over } k, l$$

$$[\,\delta^{ij} = 1 \quad i=j, \qquad 0 \quad i\neq j\,]$$

$$\epsilon^{ijm}\,\epsilon^{klm} = (\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \qquad \text{sum over } m$$

It is often convenient to work with ϵ^{ijk} algebraically, and I will do this from time to time in this course.

The curl is related to an integral quantity called the circulation.

$$\mathcal{C} = \oint_C d\vec{x}\cdot\vec{\omega}$$

To see the differential relation, compute the circulation about a small square oriented $\perp$ to $\hat{x}$

compute the loop integral in the right-hand sense about $\hat{x}$

$$\begin{aligned} C_x \cong & - a\,\omega_y(\vec{x} + \tfrac{a}{2}\hat{z}) \\ & + a\,\omega_y(\vec{x} - \tfrac{a}{2}\hat{z}) \\ & + a\,\omega_z(\vec{x} + \tfrac{a}{2}\hat{y}) - a\,\omega_z(\vec{x} - \tfrac{a}{2}\hat{y}) \end{aligned}$$

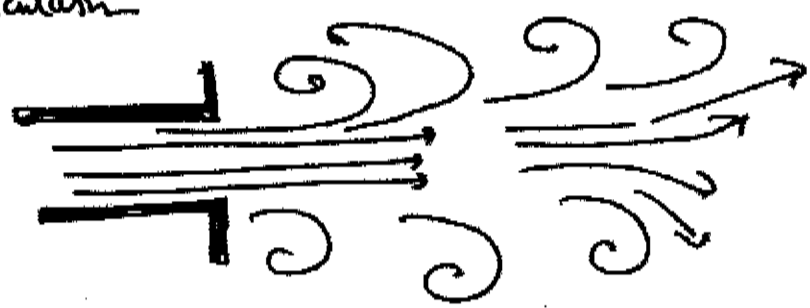
$$C_x \cong a^2\left(\frac{\partial}{\partial y}\omega_z - \frac{\partial}{\partial z}\omega_y\right)$$

The general relation – called Stokes' theorem – is

$$\int_S d^2x \; \hat{n}\cdot(\vec{\nabla}\times\vec{w}) = \int_C d\vec{x}\cdot\vec{w}$$

for any differentiable vector field and for any surface S bounded by a curve C (taken in the right-hand sense)

If $\vec{\nabla}\times\vec{w} = 0$, the circulation about any curve vanishes. Such a flow is called irrotational. Typical fluid flows are not irrotational; they have vortices, small cycles with nonzero circulation

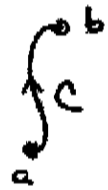


However, irrotational flow is a convenient first approximation for many fluid problems. An irrotational, incompressible flow satisfies

$$\rho = \underline{\text{const}} \qquad \vec{\nabla}\cdot\vec{v} = 0 \, , \quad \vec{\nabla}\times\vec{v} = 0$$

I assume that you have seen more general proofs of Stokes' theorem and the divergence theorem in your math classes. It is interesting to put together these two theorems and also the fundamental theorem of the calculus into a set of three integral theorems:

$$\int_C d\vec{x}\cdot\vec{\nabla}\phi = \phi(b) - \phi(a)$$



$$\int_S d^2x\,\hat{n}\cdot(\vec{\nabla}\times\vec{\omega}) = \int_C d\vec{x}\cdot\vec{\omega}$$

$$\int_V d^3x\,\vec{\nabla}\cdot\vec{\omega} = \int_S d^2x\,\hat{n}\cdot\vec{\omega}$$

On the left-hand side, we have $\vec{\nabla}$ acts on a field, integrated over a manifold M. On the right-hand side, the field is integrated over the boundary of M (∂M). These are all special cases of a grand theorem, also called <u>Stokes' theorem</u>. I'll give you a fancy proof at an appropriate point later in the course.