Solution of Laplace's Equation by Conformal Mapping

Dec. 6

In these last two lectures, I would like to discuss some more advanced aspects of the mathematical theory of electro- and magneto-statics. In this lecture, I will discuss a special method for the solution of Laplace's equation in two dimensions.

A point in the plane (x,y) can be represented by a complex number

$$z = x + iy$$

Write also

$$\bar{z} = x - iy \qquad \text{("complex conjugate" of } z\text{)}$$

Any function of x and y can be written equally well as a function of z and $\bar{z}$, since

$$x = \frac{z+\bar{z}}{2} \qquad y = \frac{-i}{2}(z-\bar{z})$$

Typically, a real-valued function must depend on both z and $\bar{z}$. A complex-valued function, however, could depend on z or $\bar{z}$ only. For example

$$f(z) = z^2$$

is real-valued on the x and y axes but complex-valued at a general point $z = x + iy$: $z^2 = x^2 - y^2 + 2ixy$. Write $\mathrm{Re}\, f(z)$ and $\mathrm{Im}\, f(z)$ for the real and imaginary parts of f. In this case

$$\mathrm{Re}\, f = x^2 - y^2 \qquad \mathrm{Im}\, f = 2xy$$

It is curious that:

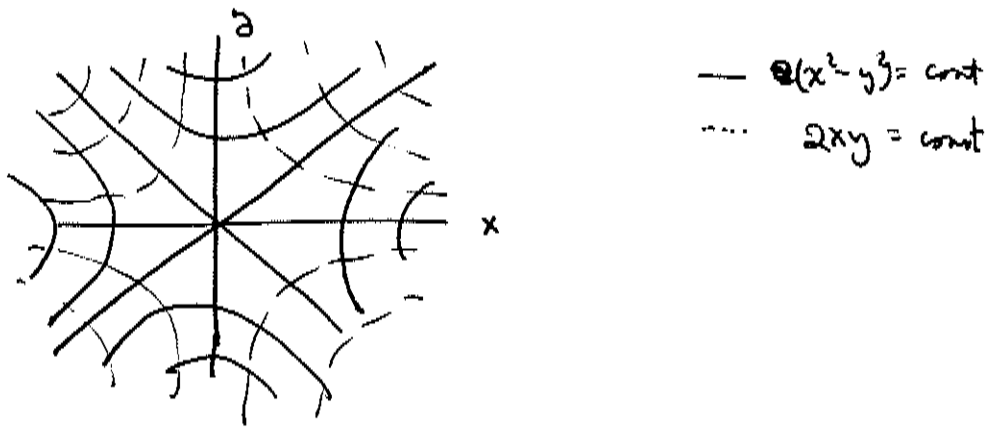
① Both $\mathrm{Re} f$ and $\mathrm{Im} f$ satisfy Laplace's equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mathrm{Re} f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \mathrm{Im} f = 0$$

② The gradients of $\mathrm{Re} f$ and $\mathrm{Im} f$ form two mutually orthogonal vector fields:

$$\vec{\nabla}(\mathrm{Re} f) = (2x, -2y) \qquad \vec{\nabla}(\mathrm{Im} f) = (2y, 2x)$$

so that the lines of constant $\mathrm{Re} f$ are parallel to the gradient of $\mathrm{Im} f$, and vice versa:



I will now prove that these properties are shared by every complex-valued function $f(z)$ which depends on z but not on $\bar{z}$. Such a function is called an analytic function of z. Analytic functions have many other interesting properties. For example, if $f(z)$ is analytic and has no singularities in a region R, then the Taylor series of $f(z)$ about $z_0 \in R$ has a finite radius of convergence, and

$$\oint_C dz\, f(z) = 0$$

for a closed contour C entirely contained in R. However, the properties ① and ② on the previous page are sufficiently interesting for one lecture!

Let's first verify property ①. If $\frac{\partial}{\partial \bar z} f = 0$,

$$0 = \frac{\partial}{\partial \bar z} f = \frac{\partial x}{\partial \bar z}\frac{\partial}{\partial x} f + \frac{\partial y}{\partial \bar z}\frac{\partial}{\partial y} f$$

$$= \frac{1}{2}\frac{\partial}{\partial x} f + \frac{i}{2}\frac{\partial}{\partial y} f \qquad \text{using the eqs. for } x \text{ and } y \text{ on p. 1}$$

The real part of this equation is:

$$\frac{\partial}{\partial x}\mathrm{Re} f - \frac{\partial}{\partial y}\mathrm{Im} f = 0$$

The imaginary part is

$$\frac{\partial}{\partial x}\mathrm{Im} f + \frac{\partial}{\partial y}\mathrm{Re} f = 0$$

These relations are called the Cauchy-Riemann equations. Then

$$\vec\nabla(\mathrm{Re} f) = \left(\frac{\partial}{\partial y}\mathrm{Im} f,\ -\frac{\partial}{\partial x}\mathrm{Im} f\right)$$

which is orthogonal to $\vec\nabla(\mathrm{Im} f)$. Now take $\frac{\partial}{\partial x}$ of the first equation:

$$\frac{\partial^2}{\partial x^2}(\mathrm{Re} f) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathrm{Im} f = \frac{\partial}{\partial y}\left(-\frac{\partial}{\partial y}\mathrm{Re} f\right)$$

so

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathrm{Re} f = 0 \qquad \text{and similarly for } \mathrm{Im} f.$$

This verifies ①.

Let's apply this principle to some simple problems of electrostatics, and then to some not-so-simple ones. A point charge in two-dimensions has the electrostatic potential (like a line source in 3-dimensions)

$$\Phi(x,y) = \frac{Q}{2\pi\epsilon_0}\log r = \frac{Q}{2\pi\epsilon_0}\log[(x^2+y^2)^{1/2}]$$

This is the real part of

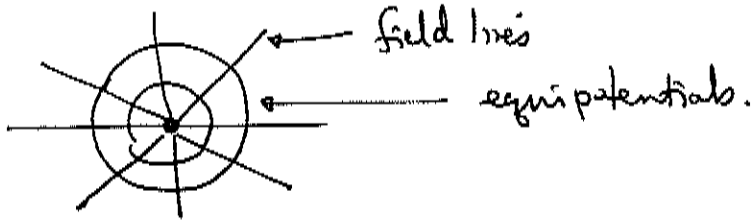
$$f(z) = \frac{Q}{2\pi\epsilon_0}\log z$$

In terms of cylindrical coordinates r, ϕ $\qquad \log z = \log r + i\phi$,

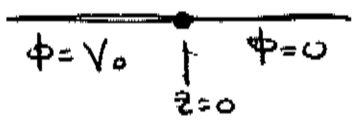
so

$$\operatorname{Im} f = \frac{Q}{2\pi\epsilon_0}\phi$$

The lines of constant $\operatorname{Im} f$ are the field lines

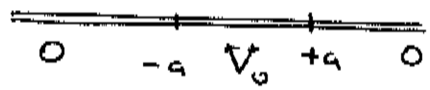


Actually, this function gives us the solution to another simple problem: Find a solution to Laplace's equation for $y > 0$ with the Dirichlet boundary condition



The solution is $\qquad \varphi = \dfrac{V_0}{\pi}\phi$

which solves Laplace's equation because it is the Im part of an analytic function. Similarly, the solution with the boundary conditions

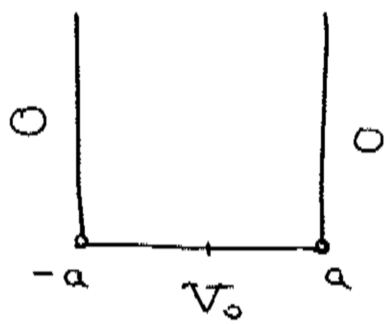


is

$$\varphi = \operatorname{Im} \frac{V_0}{\pi} \left\{ \log(z-a) - \log(z+a) \right\}$$

$$= \frac{V_0}{\pi} \left(\tan^{-1} \frac{y}{x-a} - \tan^{-1} \frac{y}{x+a} \right)$$

Now, what if the given boundary conditions are:



Here is a strategy: Find an analytic function $\omega(z)$ which maps the interior of this figure to the upper half-plane, mapping $z = \pm a$ to $\omega = \pm a$. An analytic mapping is also called a conformal mapping, because it preserves angles. Then

$$\varphi = \operatorname{Im} \left\{ \frac{V_0}{\pi} \log\left(\frac{\omega(z)-a}{\omega(z)+a} \right) \right\}$$

is a solution to Laplace's equation which satisfies the Dirichlet boundary conditions. Hence, it is the unique solution.

It is not hard to find the appropriate mapping.

Consider

$$\omega(z) = a \sin \frac{\pi z}{2a}$$

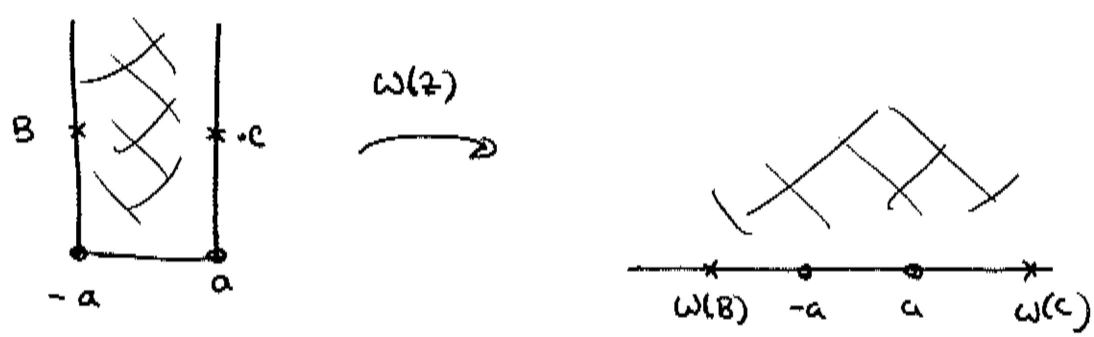
for $z = x$ (real), $-a < x < a$ this function maps the interval $(-a, a)$ onto $(-a, a)$. At $z = a$, $\sin \frac{\pi z}{2a} = \sin \frac{\pi}{2} = 1$.

What happens to $\sin(\frac{\pi z}{2a})$ after it hits 1? If we continue to higher x, the sin function comes back down and retraces its path from a to $(-a)$. But, if we set $z = a + iy$

$$\begin{aligned}
\omega &= a \sin \frac{\pi}{2a}(a+iy) \\
&= a \sin\left(\frac{\pi}{2} + i\frac{\pi y}{2a}\right) \\
&= a \cos\left(i\frac{\pi y}{2a}\right) \\
&= a \cosh\left(\frac{\pi y}{2a}\right)
\end{aligned}$$

and, as y goes from 0 to ∞, this goes from a to ∞.

Thus:



The solution is then

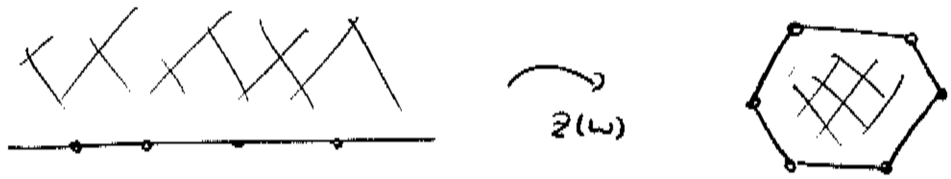
$$\varphi = \mathrm{Im}\left\{ \frac{V_0}{\pi} \log \frac{\left(\sin \frac{\pi z}{2a} - 1\right)}{\left(\sin \frac{\pi z}{2a} + 1\right)} \right\}$$

where "sin" here is the complex sin function

$$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

$$= \sin x \cosh y + i \cos x \sinh y .$$

It turns out that one can write a formula for a function $z(w)$ that maps the upper half plane into any polygon:



This is called the Schwarz-Christoffel transformation. Then the inverse function $w(z)$ gives the solution to the electrodynamics problem with boundary conditions on this polygon.

A useful type of conformal mapping is the fractional linear transformation

$$w = \frac{az+b}{cz+d} \qquad a, b, c, d = \text{const.}$$

Since $$w = \frac{a}{c} - \left(\frac{ad-bc}{c}\right)\frac{1}{cz+d}$$

an arbitrary fractional linear transformation can be built up as a sequence of translations: $w = az + b$ and inversions $w = 1/z$. These transformations map circles into circles. This is obvious for translations, for inversions it requires a computation:

An inversion carries

$$(x,y) = z \longrightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2+y^2}, \text{ so}$$

$$(x,y) \to (x',y') \qquad x' = \frac{x}{x^2+y^2} \qquad y' = \frac{-y}{x^2+y^2}$$

now if x, y lie on a circle of radius R with center at (a,b)

$$(x-a)^2 + (y-b)^2 = R^2$$

$$x^2+y^2 - 2ax - 2by + a^2 + b^2 = R^2$$

so

$$(R^2-a^2-b^2)\frac{1}{x^2+y^2} + 2a\frac{x}{x^2+y^2} + 2b\frac{y}{x^2+y^2} = 1$$

$$\frac{1}{x^2+y^2} + 2\frac{x}{x^2+y^2}\left(\frac{a}{R^2-a^2-b^2}\right) + 2\frac{y}{x^2+y^2}\left(\frac{b}{R^2-a^2-b^2}\right) = \frac{1}{R^2-a^2-b^2}$$

$$(x')^2+(y')^2 + 2(x')\left(\frac{a}{R^2-a^2-b^2}\right) - 2(y')\left(\frac{b}{R^2-a^2-b^2}\right) = \frac{1}{R^2-a^2-b^2}$$

$$\left(x' + \frac{a}{R^2-a^2-b^2}\right)^2 + \left(y' - \frac{b}{R^2-a^2-b^2}\right)^2 = \frac{1}{R^2-a^2-b^2} + \frac{a^2+b^2}{(R^2-a^2-b^2)^2}$$

$$= \frac{R^2}{(R^2-a^2-b^2)^2}$$

so (x',y') lie on a circle of radius $R/|R^2-a^2-b^2|$.

If $(R^2-a^2-b^2)=0$ we find

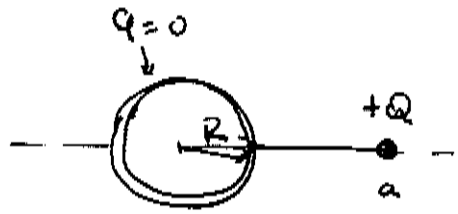
$$2ax' - 2by' = 1 \qquad \text{a line}$$

similarly, $z \to \frac{1}{z}$ carries lines into circles. More generally,

$$z \to \frac{az+b}{cz+d}$$

carries circles into circles, where a line is a degenerate case of a circle.

Here is a relatively easy problem solved by the fractional linear transformation: Find φ for the situation:



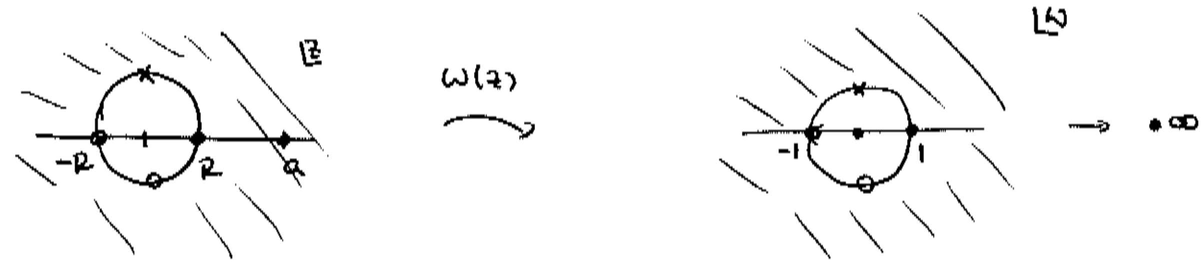
We know that this problem can be solved by the method of images. But we can also solve it by finding a fractional linear transformation that carries the three points

$$z = -R,\ R,\ a \quad \text{to} \quad \omega = -1,\ 1,\ \infty$$

Since the fractional linear transformation has 3 free parameters, there is a fractional linear transformation that carries any three chosen points to three arbitrary locations in the plane. In this case, the transformation we need is:

$$\omega = \frac{az - R^2}{(a-z)}$$

this carries:



In the ω plane, a solution with $\varphi = 0$ on the circle is

$$\varphi = \mathrm{Re}\left[a \log \frac{w}{R}\right] \qquad a = \text{const}$$

so the solution to our original problem is

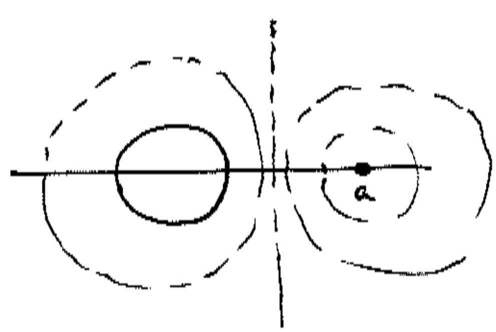
$$\varphi = \mathrm{Re}\; a \log \frac{az - R^2}{R(a-z)}$$

since the potential of a point charge is $\varphi = \frac{Q}{2\pi\epsilon_0} \log r$

we can determine the overall constant a, to find

$$\varphi = \frac{Q}{2\pi\epsilon_0}\left(\log|x - \vec{x}_a| - \log|x - \vec{x}_i| - \log a/R\right)$$

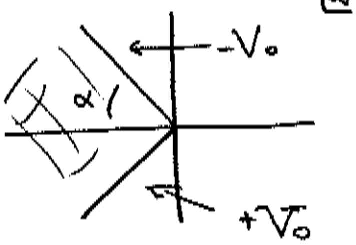
where $\vec{x}_a = (a, 0)$ and $\vec{x}_i = (R^2/a, 0)$ is the image charge location. Notice that a circle about 0 in the ω plane is an equipotential in the z plane:



so this method also solves problems in which φ obeys the

Dirichlet bndy conditions $\varphi = \text{const}$ on two circles.

If we start from



$$\varphi = \mathrm{Im}\left[\frac{V_0}{\alpha}(\log z - i\pi)\right]$$

and perform a fractional linear transformation that carries:

$$0 \leftrightarrow R \qquad \infty \leftrightarrow -R$$

we can solve a problem with Dirichlet bndy conditions on two circular arcs that meet at an angle 2α:

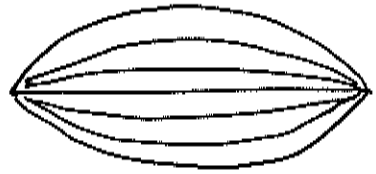
the required transformation is

$$w(z) = c\left(\frac{R-z}{R+z}\right) \qquad c \text{ real}$$

so

$$\varphi = \frac{V_0}{\alpha}\,\mathrm{Im}\left\{\log\left(\frac{R-z}{R+z}\right) - i\pi\right\}$$

The lines of constant φ look like:



These are also circular arcs.

This function also solves a problem in fluid mechanics.

Consider an incompressible fluid: $\vec{\nabla}\cdot\vec{V} = 0$ in two dimensions. If the flow is also irrotational $\vec{\nabla}\times\vec{V} = 0$, $\vec{V} = -\vec{\nabla}\chi$, where χ is called the velocity potential, and χ obeys the Laplace equation. Then

$$\chi = \mathrm{Re}\, f(z) \qquad \text{for some } f,$$

and if we set $\psi = \mathrm{Im}\, f(z)$, the lines of constant ψ are the lines parallel to $\vec{\nabla}\chi$. Thus, the fluid flows along these lines, called "streamlines". In the lens-shaped region on the previous page, the arcs are streamlines of a flow that goes from one corner to the other.

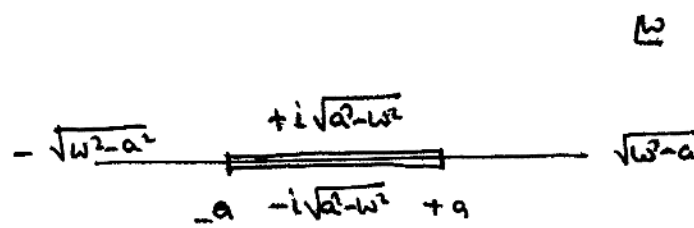
With this in mind, consider the Joukowsky transformation

$$z = \omega + \sqrt{\omega^2 - a^2} \qquad \omega = \tfrac{1}{2}\left(z + \frac{a^2}{z}\right)$$

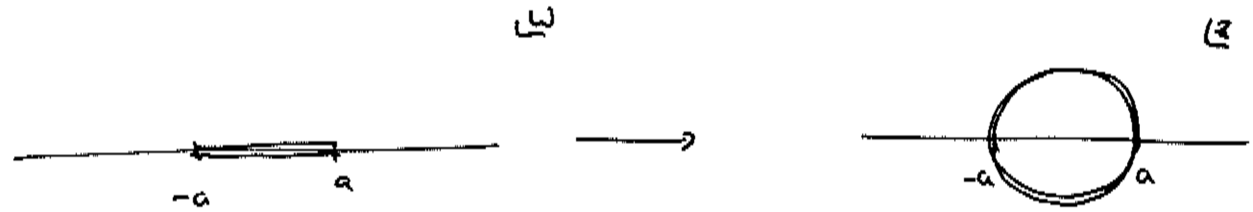
Near $z = \infty$, we have $z \approx 2w$, a regular mapping. The real ω axis for $|\omega| > a$ is mapped to the real z axis for $|z| > a$. However, for $\omega < a$, the square root is

$$\sqrt{\omega^2 - a^2} = \pm i\sqrt{a^2 - \omega^2}$$

depending on whether ω is taken with a small positive or negative imaginary part:

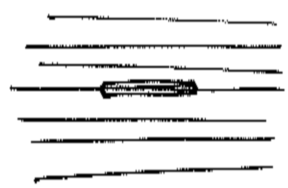


Thus, the lines just above and just below the real axis in the ω plane for $|\omega| < a$ map to a circle in the z plane



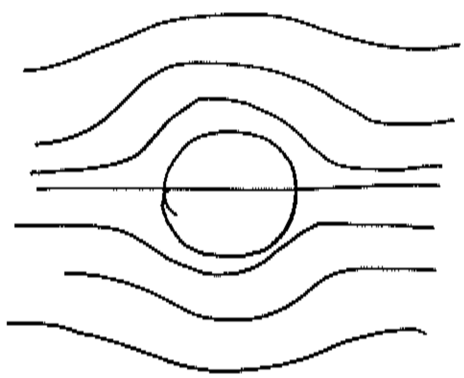
a flow in the ω plane is described by the streamlines

$$\psi = \operatorname{Im} \omega$$



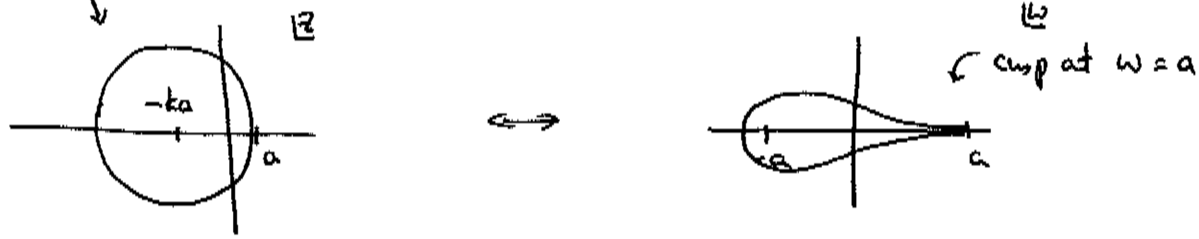
so the streamlines

$$\psi = \operatorname{Im} \omega(z) = \operatorname{Im} \left(z + \frac{a^2}{z} \right)$$



describe an incompressible, irrotational flow around a long cylinder. Given this solution we can find some more exotic applications. The Joukowski transformation also maps.

$$\left|\frac{z}{a}+k\right| = 1+k$$



$$\left|\frac{z}{a} + i\lambda + k(i\lambda+1)\right| = (1+k)\sqrt{\lambda^2+1}$$

which begins to resemble an air-foil !

This gives a taste of some more exotic methods ~~that~~ can be applied to solve Laplace's equation in 2 dimensions. For more examples, see Carrier, Krook, and Pearson, Functions of a Complex Variable.