The Energy in a Magnetic Field

Dec. 1

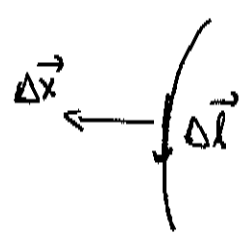
In the previous lecture, we noted that an electric current moving through a magnetic field feels a force

$$\int d\vec{F} = \int d^3y \, \vec{J} \times \vec{B} \underset{\substack{\text{for a} \\ \text{wire}}}{=} I \int d\vec{l} \times \vec{B}$$

This means that a current loop carrying a fixed current I gets sucked into a region with a large B field

Thus, one might assume that moving the loop into the field lowers the energy of the configuration. But this is odd, because the current is oriented in such a way that it increases the strength of the $\vec{B}$ field.

Let's analyze all of the work done when the current loop moves into the field. First of all, let $\Delta \vec{l}$ be an element of the loop, and let $\Delta \vec{x}$ be a motion of this element in a time Δt

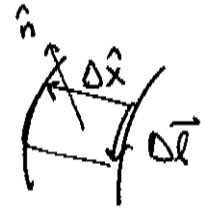


The work done on the charges moving in the wire by the $\vec{B}$ field is:

$$\Delta W_1 = \Delta\vec{x} \cdot (I\,\Delta\vec{\ell} \times \vec{B})$$

$$= I\,(\Delta\vec{x} \times \Delta\vec{\ell}\,) \cdot \vec{B}$$

Now $\Delta\vec{x} \times \Delta\vec{\ell}$ is a unit of area swept out by the motion of the loop. So



$$\Delta W_1 = I\,\Delta\vec{a} \cdot \vec{B}$$

and so the total over the loop is

$$\Delta W_1 = I\,\Delta\left(\int d^2x\,\hat{n}\cdot\vec{B}\right) = I\,\Delta\Phi_S$$

where Φ_S is the flux of $\vec{B}$ through the loop.

However, this is not the whole story. If we move the loop in the direction of $\Delta\vec{x}$, the charges in the loop also have a velocity $\vec{w} = \Delta\vec{x}/\Delta t$. This leads to another magnetic force

$$\vec{F}_i = q_i\,\vec{w} \times \vec{B}$$

on each charge. Summing over the whole set of charges on the wire

$$\vec{F} = n\,|\Delta\vec{\ell}| \cdot q\,\vec{w} \times \vec{B}$$

The charges are moving with velocity $\hat{\Delta\ell}\,v$, so this force also does work:

$$\frac{\Delta W_2}{\Delta t} = \hat{\Delta\ell} \cdot v \cdot n\,|\Delta\vec{\ell}|\,q\,\vec{w} \times \vec{B}$$

or, since $I = qnv$

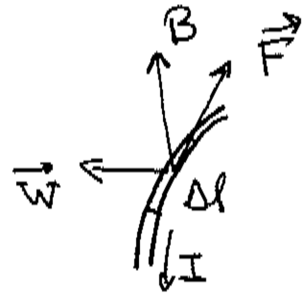
$$\Delta W_2 = I \; \vec{\Delta l} \cdot (\vec{\Delta x} \times \vec{B}) = -\Delta W_1$$

so in all, the magnetic forces do no work on the charges. This is obvious, because the total force is normal to the total velocity, but it is good to see how the pieces come together.

But there is one more bit of work we should account. The force associated with moving the wire

$$\Delta \vec{F} = \frac{I|\vec{\Delta l}|}{v} \; \vec{w} \times \vec{B}$$

has a component along the wire:

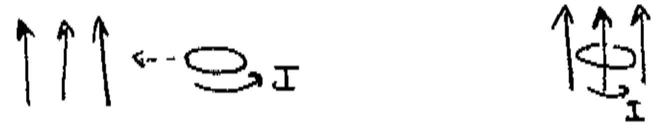


Notice that, in the figure, $\vec{F}$ points in such a way as to impede the flow of current. Then, in order to maintain a constant current, the electrostatic potential in the wire must increase to compensate this force. Another way of saying this is that the battery that drives the current I must do extra work, just equal and opposite to ΔW_2

so

$$\Delta W_{\text{battery}} = -\Delta W_2 = I\,\Delta\Phi_S$$

When we move the current loop into a $\vec{B}$ field with the same sense



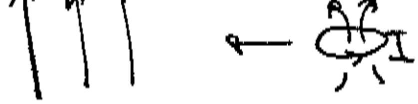
the battery does positive work equal to $I\,\Phi_S$, over and above the work done against the resistance in wire. When we move the current loop out, the battery does work equal to $-I\,\Phi_S$

all this energy is recovered. So it makes sense to associate the energy

$$E = I\,\Phi_S = I\int_S d^2x\,\hat{n}\cdot\vec{B}$$

with the energy stored by the loop in the magnetic field.

It is useful to take note of the sign of this energy. If we move a loop which creates a magnetic field into a magnetic field of the same sense



the magnetic force opposes the current and the battery must do positive work. If we move the loop into a magnetic field of the opposite sense,

the magnetic force assists the current and the battery does negative work. The ~~statement that~~ electromagnetic forces oppose a motion that tends to make $\vec{B}$ stronger is called Lenz's Law.

Replacing $\vec{B}$ with $\vec{A}$, we can rewrite the energy of a current distribution in a magnetic field as

$$E = \oint_{C=\partial S} I\, d\vec{l}\cdot\vec{A} = \int d^3y\ \vec{J}(y)\cdot\vec{A}(y)$$

If we represent a general stationary current distribution as a large number of small current loops, we see ~~that~~ this is a general expression. In this equation, though, $\vec{J}$ is a fixed current distribution in an external $\vec{B}$ field. What if we want the total energy of an array of currents that interact with one another through the $\vec{B}$ fields that they create?

To analyze this system, use the trick we used earlier in electrostatics: Bring each little increment of current in separately from infinity. Let $\alpha \in [0,1]$ parametrize this process. At the stage α, the increment of energy is

$$dE = \int d^3y \, (d\alpha \, \vec{J}(y)) \, (\vec{A}(y))_{\alpha \vec{J}}$$

since $\vec{A}$ due to $\alpha\vec{J}$ is α times the final A, the total energy is

$$\int dE = \int d\alpha \int d^3y \, \vec{J}(y) \cdot \alpha \vec{A}(y)$$

$$E = \frac{1}{2} \int d^3y \, \vec{J}(y) \, \vec{A}(y)$$

There is one more very pretty simplification. Note that $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$.
Then

$$E = \frac{1}{2\mu_0} \int d^3y \, (\vec{\nabla} \times \vec{B}) \cdot \vec{A}$$

integrate by parts $= \frac{1}{2\mu_0} \int d^3y (+\vec{B} \times \overleftarrow{\nabla} \cdot \vec{A}) = \frac{1}{2\mu_0} \int d^3y \, \vec{B} \cdot \vec{\nabla} \times \vec{A}$

or

$$E = \frac{1}{2\mu_0} \int d^3y \, |\vec{B}|^2 \quad !$$

This has just the same form as the energy of an electrostatic field, $E = \frac{\epsilon_0}{2} \int d^3y \, |\vec{E}|^2$.

In the presence of magnetic media, these formulae have a simple generalization. We would like to suppress the bound currents and deal only with the free currents. If we move currents in from infinity, these must be free currents, that is, currents under our explicit control. Once you keep this in mind, very little else changes. The energy of a free current loop in an external magnetic field is

$$E = \int d^3y \ \vec{J}_f \cdot \vec{A}$$

The energy associated with the magnetic fields of an assemblage of free currents is [as long as we are dealing with linear media!]

$$E = \frac{1}{2} \int d^3y \ \vec{J}_f \cdot \vec{A}$$

Now, in the presence of magnetic media $\vec{\nabla} \times \vec{H} = \vec{J}_f$, so this is just equal to

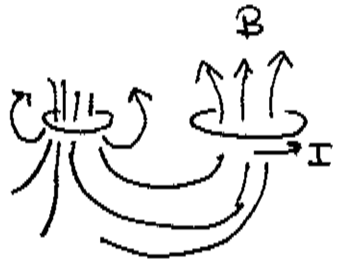
$$E = \frac{1}{2} \int d^3y \ \vec{H} \cdot \vec{B} \qquad \text{[linear media!]}$$

which is the precise analogue of the electrostatic expression $E = \frac{1}{2} \int d^3y \ \vec{D} \cdot \vec{E}$. This expression for the energy divides into pieces

$$E = \frac{1}{2\mu_0} \int d^3y \ B^2 - \frac{1}{2} \int d^3y \ \vec{M} \cdot \vec{B}$$

The extra work due to the medium is negative for a paramagnet, since the atomic magnets lower their energy when they orient with the field, and is positive for a diamagnet. The latter conclusion accords with Lenz's law.

Now, all of these calculations are correct, but there is something strange about them. Throughout this lecture, whenever I have manipulated a current, I have moved it into a pre-existing magnetic field. But, actually, if we have a configuration



$$E = I \int d^2x\, \hat{n}\cdot\vec{B}$$

we could have built it up in three different ways:

1.) Move the loop on the right into the field.

2.) Move the loop on the left close to the loop on the right

3.) Start with zero current in the loop on the left, and turn the current up, increasing $\vec{B}$.

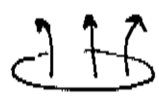
In methods 2 and 3, the current loop on the right stays still. We do not yet have any equation that requires work to be expended in these two cases.

Michael Faraday found this situation bizarre. So he experimented with all three methods and found that, in each case, the extra work done by the battery attached to the current loop on the right is identical

$$W_{battery} = I \, \Delta\Phi_s$$

this is Faraday's law (version 1). In the case (1) this work is accounted for by a magnetic force, but in cases (2) and (3) there is no such explanation. Instead, we must insist that there is an electric force which is opposed by the battery:

increasing flux of $\vec{B}$ $\longrightarrow$ nonzero $\oint d\vec{l} \cdot \vec{E}$



Since the rate of work done to oppose this $\vec{E}$ field is

$$\frac{dW_{batt.}}{dt} = - I \oint d\vec{l} \cdot \vec{E}$$

we see that the formula

$$\oint d\vec{l} \cdot \vec{E} = - \frac{d\Phi_s}{dt}$$

gives the $\vec{E}$ field with correctly produces $W_{battery}$ for any method in which $\vec{B}$ changes. This equation is also called Faraday's law (version 2)

$$\frac{d}{dt}\int_S d^2x\ \hat{n}\cdot\vec{B} = -\int_{C=\partial S} d\vec{l}\cdot\vec{E}$$

This is the integral form of the relation (also, Faraday's law)

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

As before, the sign of $\vec{E}$ is such as to oppose an ~~[illegible]~~ increase in magnetic flux — Lenz's Law! The creation of an electric field as the result of a change in a magnetic field is called "magnetic induction".

Faraday's law is the first step toward formulating a theory of time-dependent electric and magnetic fields. We have so far:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

It turns out that these equations are inconsistent!

Maxwell realized that a consistent set of equations can be obtained by the addition of one more term. This gives "Maxwell's equations", a time-dependent theory of electricity and magnetism in which the fields actually take off from the charges and carry on in their own right. We'll study this dynamic theory in great detail in Physics 121.