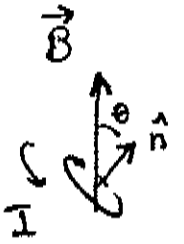
Magnetic Media - 2

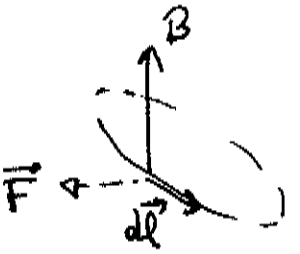
Nov. 29

Now that we have worked out the kinematics of magnetism in matter, it is time to discuss the physics. What is the origin of magnetic moments in matter? Why do they form?

To begin, I would first like to point out that if an atom has a magnetic moment which is free to rotate, it can be oriented by a magnetic field. To see this, let's compute the force on a magnetic dipole due to a magnetic field. As a simple model, think about a current loop of radius a, oriented at an angle θ with respect to a $\vec{B}$ field:



In cylindrical coordinates $\hat{r}\ \hat{\phi}\ \hat{z}$ about the B field, the force on a current element $Id\vec{l}$

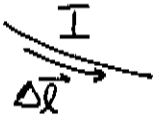


points in the $\hat{r}$ direction ($\perp$ to $\vec{B} \parallel \hat{z}$ and $\hat{\phi}$). The current

is supplied by a density of particles n (particles/m) with charge q, moving with average velocity v in the direction $d\vec{l}$ of the loop

$$I = qnv \qquad (C/sec)$$

The force on a small element of the loop



$$\Delta\vec{F} = \sum_i q_i v_i \, \Delta\hat{l} \times \vec{B}$$

we can replace $\sum_i v_i = n|\Delta\vec{l}|\, v$

so

$$\Delta\vec{F} = qvn \, \Delta\vec{l} \times \vec{B}$$

or, for a whole loop

$$\oint d\vec{F} = \oint I d\vec{l} \times \vec{B}$$

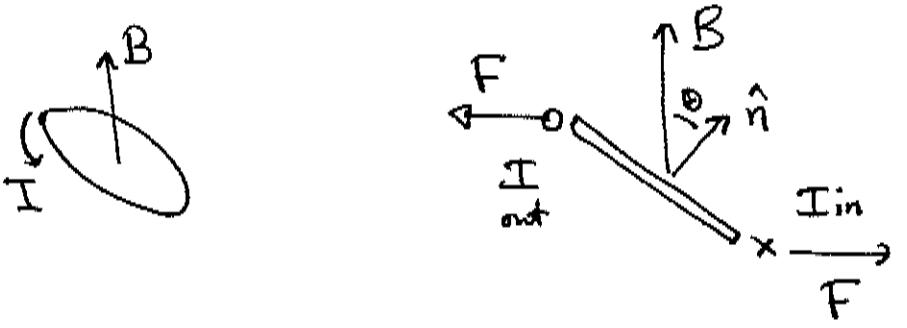
similarly, if we have a volume current density $\vec{j}$

$$\int d\vec{F} = \int d^3y \; \vec{j}(y) \times \vec{B}$$

For a small loop, the total force is zero, since

$$\oint d\vec{l} = 0$$

however the forces are unbalanced:

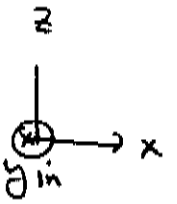


There is a torque which tends to align the dipole with the magnetic field. It is not hard to compute this torque explicitly:

If

$$\vec{B} = (0,0,B) \qquad \hat{n} = (\sin\theta, 0, \cos\theta)$$

normal to loop.



and we measure ϕ from the highest part in the loop

$$d\vec{R}(\phi) = R\,d\phi\,(\sin\phi\cos\theta, -\cos\phi, -\sin\phi\sin\theta)$$

$$d\vec{F} = IRd\phi \cdot B\,(-\cos\phi, -\sin\phi\cos\theta, 0)$$

$$\vec{r} = R\,(-\cos\phi\cos\theta, \sin\phi, \cos\phi\sin\theta)$$

$$\vec{r}\times d\vec{F} = IR^2B\,d\phi\,(\sin\phi\cos\phi\sin\theta, -\cos^2\phi\sin\theta, \sin\phi\cos\phi(\cos^2\theta+1))$$

integrating over $\int d\phi$ the terms with $\sin\phi\cos\phi$ cancel!

$$\int \vec{r}\times d\vec{F} = \text{total } \vec{\tau} = IR^2\cdot\pi\cdot B\cdot(-\hat{y})$$

$$= I\pi R^2(\hat{n}\times\vec{B})$$

and we recognize the magnetic dipole moment

$$\vec{m} = I\pi R^2 \hat{n}$$

Actually, it is not difficult to show that these results are completely general for dipole distributions. The total force on a current distribution is

$$\vec{F} = \int d^3y\, \vec{J} \times \vec{B}$$

But we proved that, for any steady current $\int d^3y\, \vec{J}(y) = 0$

The torque on a current distribution is

$$\vec{\tau} = \int d^3y\, \vec{y} \times (\vec{J} \times \vec{B})$$

$$= \int d^3y \left[\vec{J}\,(\vec{y}\cdot\vec{B}) - \vec{B}\,(\vec{y}\cdot\vec{J}) \right]$$

But we also proved that

$$\int d^3y\, (y^i J^j + y^j J^i) = 0$$

so

$$\int d^3y\, \vec{y}\cdot\vec{J} = 0$$

$$\int d^3y\, \vec{J}\,(\vec{y}\cdot\vec{B}) = \frac{1}{2}\int d^3y \left[\vec{J}\,(\vec{y}\cdot\vec{B}) - \vec{y}\,(\vec{J}\cdot\vec{B}) \right]$$

Then the expression for $\vec{\tau}$ becomes.

$$\vec{\tau} = \int d^3y\, \tfrac{1}{2}\left\{ \vec{J}\,(\vec{y}\cdot\vec{B}) - \vec{y}\,(\vec{J}\cdot\vec{B}) \right\} = \int d^3y\, \tfrac{1}{2}(\vec{y}\times\vec{J}) \times \vec{B}$$

but this is exactly

$$\vec{\tau} = \vec{m} \times \vec{B}$$

While we are doing this calculation, we might as well work out the case of a nonuniform $\vec{B}$ field. Taylor-expand $\vec{B}$ about its value at $\vec{y} = 0$:

$$B^k(y) = B^k(0) + y^m \left(\frac{\partial}{\partial y^m} B^k\right)(0) + \cdots$$

We have already seen that the first term gives no net force. The second term gives

$$(\vec{F})^i = \int d^3y \; \epsilon^{ijk} \, j^j \, (y^m \nabla^m B^k(0))$$

Now, since the symmetric part vanishes,

$$\int d^3y \; y^m j^j = \int d^3y \; \epsilon^{mjl} \tfrac{1}{2} (\vec{y} \times \vec{l})^l$$

[To see this, check, for example $m = 1$ $j = 2$:

$$\int d^3y \; y^1 j^2 = \int d^3y \tfrac{1}{2}(y^1 j^2 - y^2 j^1) = \int d^3y \; \epsilon^{123} \tfrac{1}{2} (\vec{y} \times \vec{j})^3 .]$$

Inserting this back into the expression for $\vec{F}$:

$$\vec{F}^i = \int d^3y \; \underbrace{\tfrac{1}{2}(\vec{y} \times \vec{j})^l} \; \underbrace{\epsilon^{ijk} \epsilon^{mjl}} \; \nabla^m B^k(0)$$

$$= (\vec{m})^l \; (\delta^{im}\delta^{kl} - \delta^{il}\delta^{km}) \; \nabla^m B^k(0)$$

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) - \vec{m}\,\vec{\nabla}\cdot\vec{B}(0)$$

and, since $\vec{\nabla}\cdot\vec{B} = 0$ everywhere,

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

It is interesting to compare these results to those for electric dipoles. For an electric dipole, we studied the potential energy and found

$$V = -\vec{P}\cdot\vec{E}$$

leading to

$$\vec{F} = \vec{\nabla}(\vec{p}\cdot\vec{E})$$

We didn't discuss the formula for torque, but this is easy to work out

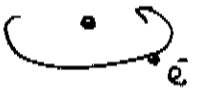
$$\vec{\tau} = \int d^3y\ \vec{y}\times\rho\vec{E} \qquad \rho = C/m^3$$

$$= \vec{p}\times\vec{E}$$

so:	electric	magnetic dipole
torque:	$\vec{p}\times\vec{E}$	$\vec{m}\times\vec{B}$
force:	$\vec{\nabla}(\vec{p}\cdot\vec{E})$	$\vec{\nabla}(\vec{m}\cdot\vec{B})$
associated V:	$-\vec{p}\cdot\vec{E}$	$-\vec{m}\cdot\vec{B}$

The similarities between the electrical magnetic cases are amazing, considering that the electric results are associated with a potential energy, while in the magnetic case this is far from obvious. We'll take up this question again in the next lecture.

In any case, we have found that if small circulating currents are present in matter, they can be oriented by magnetic fields. Where could these currents come from. Consider a single model of an atom, with an electron orbiting a nucleus



There is a circulating current here, with

$$\vec{m} = \frac{1}{2}\int d^3y\ \vec{y}\times\vec{J}$$

$$= \frac{1}{2}\vec{r}\times(-e)\vec{v} \qquad \text{(time-averaged over the orbit)}$$

actually $m\vec{r}\times\vec{v} = \vec{L}$ orbital angular momentum

so

$$\vec{m} = -\frac{e}{2m}\vec{L}$$

In quantum mechanics, $\vec{L}$ is quantized in units of

$$\hbar = \frac{h}{2\pi} \leftarrow \text{Planck's constant}$$

so a typical atomic dipole has magnitude

$$\mu_B = \frac{e\hbar}{2m_e} \qquad \text{the "Bohr magneton"}$$

$$= \begin{cases} 0.9274 \times 10^{-23} \ \text{C}\,\text{m}^2/\text{sec} \\ 0.5788 \times 10^{-5} \ \text{eV/T} \end{cases}$$

using $\text{A m}^2 = \text{J/Tesla}$

Properly harnessed, this is a lot of magnetism. Remember that

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

so $\mu_0\vec{M}$ has the units of magnetic field. Indeed, with 1 atom / cubic Å with a magnetic moment of $1\,\mu_B$, all pointing in the same direction,

$$\mu_0 M \sim \frac{\mu_0}{\frac{4\pi}{3}\kappa(10^{-10}\text{m})^3}\mu_B = 3\ \text{T}\ !$$

Before signing on, I should point out that electrons from circulating currents not only because they orbit the atomic

nucleus but also because they spin on their axes. An isolated electron is known to be always spinning, with an angular momentum

$$L = \hbar/2 \qquad \text{about some axis.}$$

Its magnetic moment is almost exactly $1\,\mu_B$. This is odd, and cannot be explained in classical terms.* The magnetic moment of an electron is, in fact

$$\mu_e = g\,\frac{e\hbar/2}{2m_e}$$

where g = "Landé g-factor" = $2\cdot(1.001159652187(4))$

(expt'l. error in last place)

Dirac's relativistic quantum theory of electrons predicts $g = 2$. Quantum electrodynamics is known to account for the (complete!) correction.

Now, how effectively can we use these magnetic moments to create a net magnetization. First of all, there has to be a net magnetic moment left over on the atom. In a covalently

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* When Goudsmit & Uhlenbeck first proposed these formulae, Lorentz used the "classical electron radius" r_e, given by $m_e c^2 = \frac{e^2}{4\pi\varepsilon_0 r_e}$, to compute the velocity at the surface of the electron. He found $v \sim 100c$ — and threw them out of his office!

bonded molecule, all electrons are paired. In the inner atomic shells of an atom, electrons are also forced to pair by the Pauli exclusion principle: All available states of spin and orbital angular momentum are filled. However, isolated atoms in a gas or impurities in a solid can have unpaired electrons, and solids made of transition metals can have unpaired electrons in partially filled d or f orbitals. Typically, these spins are unorganized. They can be oriented by a magnetic field, but they are disorganized by thermal fluctuations. In a field B, the fraction of ordered spins is roughly

$$\frac{\text{orientation energy}}{\underbrace{kT}_{\text{typical thermal energy}}} = \frac{\mu_B B}{1/40\ \text{eV}} \quad \text{at room Temperature}$$

$$= (2\times10^{-4}/T)\cdot B$$

Then the magnetization produced is of order

$$M \sim n\mu_B \cdot (2\times10^{-4}/T)\cdot \frac{B}{\mu_0} H$$

$$\sim \left[\frac{10^{-23}}{\frac{4\pi}{3}(10^{-10})^3}\left(\frac{1}{4\pi\times10^{-7}}\right)\cdot\ 2\times10^{-4}\right] H$$

$$\sim \quad [6\times10^{-4}]\ H$$

Materials with unpaired electrons that can be oriented by a magnetic field then should have $\vec{M}$ parallel to the applied field

$$\vec{M} = \chi_m \vec{H}$$

with $\chi_m > 0$; typical values of χ_m are $10^{-4} - 10^{-5}$

eg: Al 2×10^{-5} , W 8×10^{-5} ,

Liquid O_2 $(-200°C)$ 4×10^{-3}

Such materials are called paramagnetic.

In materials with no unpaired electrons, an external magnetic field can perturb an atom and mix into the atomic configuration states with nonzero angular momentum. In this case, the fraction of ordered spins is roughly

$$\frac{\mu_B B}{\text{typical atomic energy}} \sim \frac{\mu_B B}{10\,eV} \sim (0.5\times10^{-6}/T)\,B$$

Then

$$M \sim 10^{-6} H$$

The M resulting from this effect is oriented opposite to H , thus

$$M = \chi_m H$$

with $\chi_m < 0$; typical values of $|\chi_m|$ are $10^{-8} - 10^{-5}$

Au: -3×10^{-5} Ag: -2×10^{-5} H_2O: -0.9×10^{-5}

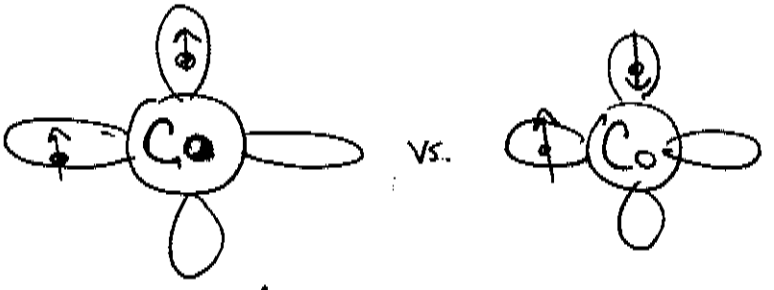
Such materials are called diamagnetic.

For either paramagnets or diamagnets, the approximate

$$B = \mu H \qquad \mu \cong \mu_0 \text{, linear relation}$$

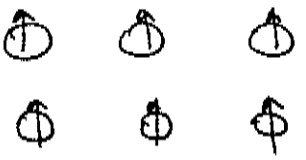
is a very good one.

In certain materials, however, unpaired spins of electrons in the d or f orbitals can become organized in a common direction. The force that leads to this organization has nothing to do with magnetic forces but rather comes from quantum mechanics. States such as



have different energies due to the interaction among electrons induced by the Pauli exclusion principle. For example, electrons with parallel spins tend to be further apart

ed thus feel less electrostatic repulsion. The energies involved are atomic (eV) scale. The spin orientation may be parallel or antiparallel on neighboring atoms. Parallel orientation leads to a ferromagnet:

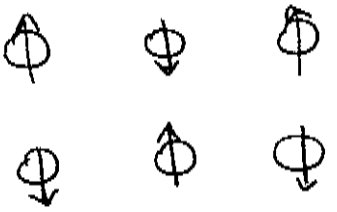


which should even have $\mu_0 M \sim$ Tesla at $H = 0$.

In fact,

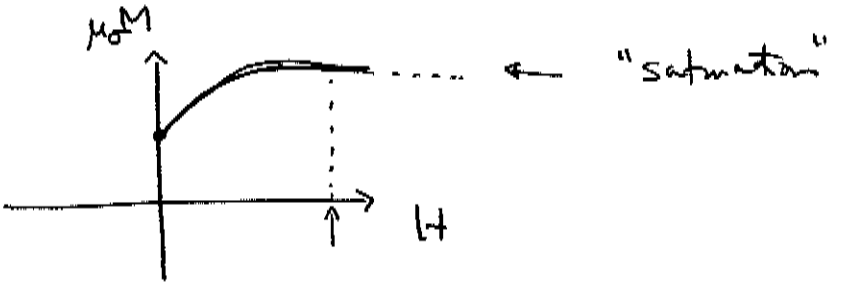
	$\mu_0 M$ at $0^\circ K$ "saturation magnetization"
Fe	1.75 T
Co	1.45 T
Ni	0.51 T

Antiparallel orientation leads to an antiferromagnet



eg. Cr, FeO, NiO, ... These materials (first recognized as such by Néel) have no macroscopic magnetism but have thermal behavior similar to ferromagnets. In both cases, the magnetic order disappears at a high temperature T_c ($\sim 500 - 1000^\circ K$).

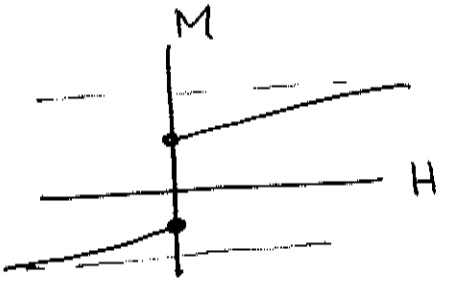
A ferromagnet has a nonlinear relation between M and H & B and H



The fields giving saturation are 10~100 gauss so $\mu_0 M \sim (10^2 - 10^3)\,\mu_0 H$. In this case

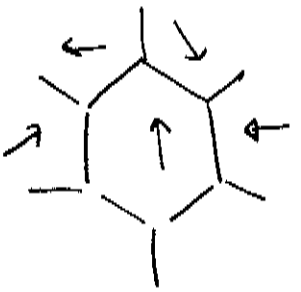
$$B \approx \mu_0 M$$

to a good approximation. In the ideal case, the direction of M follows that of H. In particular, M flips over discontinuously at $H = 0$

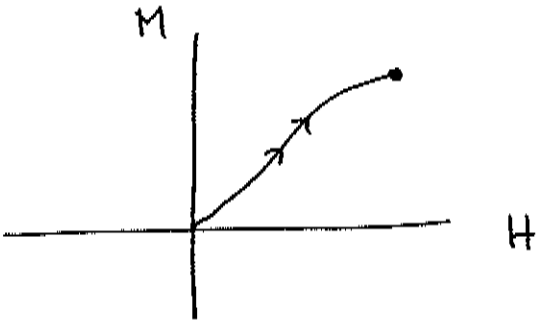


However, the actual picture of a ferromagnet is more complicated. A large piece of spin-aligned ferromagnet has a large magnetic field, and this field has substantial energy. A material can decrease this energy by breaking up

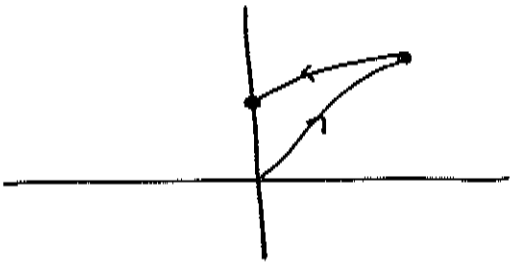
into magnetic domains, each with a different orientation



This is the typical state of a random piece of iron. Applying an external magnetic field can cause the properly oriented domains to grow at the expense of the others, leading to bulk magnetization. In this process, the iron follows a curve in the (M, H) plane



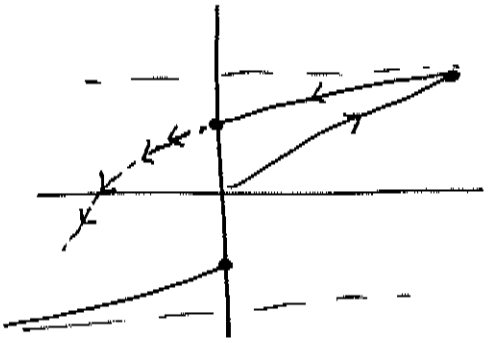
If we now turn H off slowly, the iron will relax along



some back to a different, magnetized, H = 0 state. The property that the magnetization of a ferromagnet depends

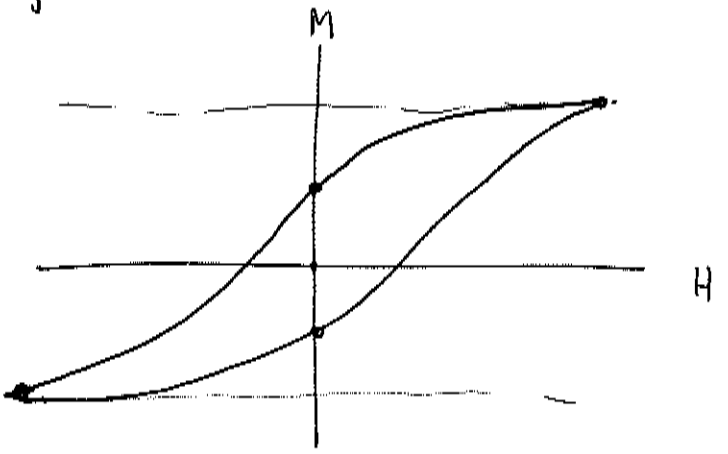
on how it has been treated is called hysteresis.

Actually, if we now let H go smoothly to negative values, the magnet will follow the curve



but at some point it will saturate in the negative direction and then follow the bottom curve when $|H|$ is decreased.

The full "hysteresis curve" has the form



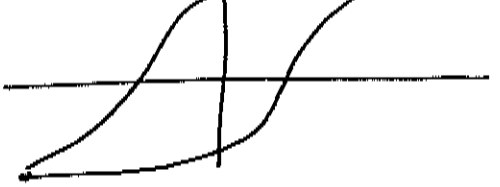
Starting from a piece of unmagnetized iron with small domains, it is possible to reach any point in the interior of this figure by successively raising and lowering H.

Now, how do we use this information to design a really good electromagnet. Any magnet uses the basic concepts

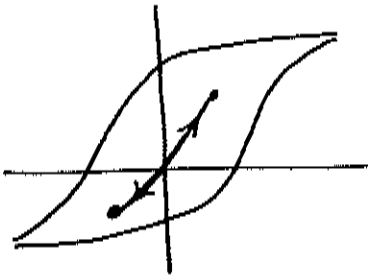
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \oint_{C=\partial S} d\vec{l}\cdot\vec{H} = I_S$$

So we want to form a "magnetic circuit" around which the flux of $\vec{B}$ can flow. And try to concentrate $\vec{H}$ in a small working region.

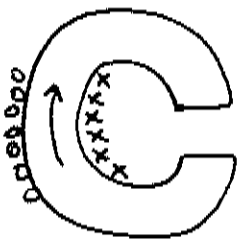
To do this, start with an iron-core solenoid. Choose an appropriate working point in the M-H plane. For high field:

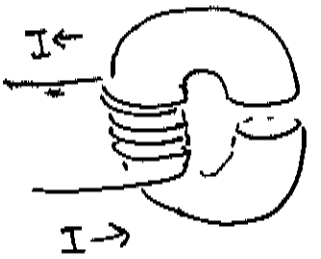


For good tunability and responsiveness (large $\partial M/\partial H$), we might stay away from saturation and work in a region where the domains are still present.

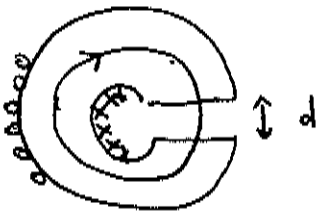


Now set up the iron to channel the $\vec{B}$ field of the solenoid.





This gives a magnetic circuit broken by a small gap



In the iron, H is very small. Since $B_\perp$ is continuous at an interface, $\mu_0 H$ in the gap is approximately equal to B in the iron. So

$$\oint d\vec{l}\cdot\vec{H} \cong H_{gap}\cdot d = IN \qquad N = \#\text{ of turns on the iron}$$

then B in the gap is given by

$$B \cong \mu_0 I \frac{N}{d}$$

and we can easily make N/d very large.