Magnetic Media

Nov. 27

In our study of electrostatics, we took account of the fact that physical media are often polarizable and thus affect the distribution of fields. In magnetostatics, we must similarly take account of the fact that physical media may be magnetized, that is, may acquire a volume density of magnetic dipoles under the influence of a magnetic field. In fact, you are familiar with many materials — ordinary iron, for example — that can have a magnetic dipole moment even if there is no applied field. This is associated with an important complication of magnetic media: Whereas in polarizable media it is easy to understand the polarizability in classical terms, magnetic properties typically cannot be understood without concepts from quantum mechanics.

Before I begin a discussion of the physical origin of magnetism in matter, let me set up the kinematics of magnetic materials. It is helpful to follow the analogy with electrostatics. In electrostatics, a polarizable medium is characterized by a density of electric dipoles

$$\vec{P} = \text{density of electric dipoles} \qquad C/m^2$$

A magnetic material is similarly characterized by a density of magnetic dipoles

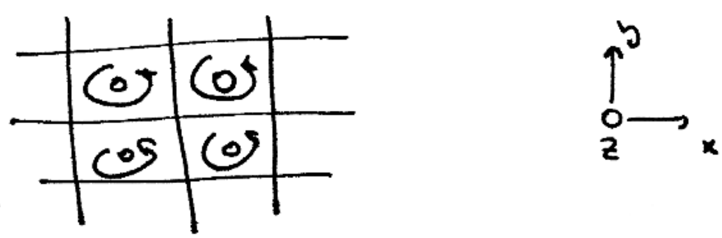
$$\vec{M} = \text{density of magnetic dipoles}$$

$$\text{units:} \quad [C/\text{sec}\cdot m^2]/m^3 = C/m\text{-sec}.$$

If $\vec{P}$ is non-uniform, there is a macroscopic charge density

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

Similarly, a nonuniform $\vec{M}$ corresponds to a macroscopic current density. To see this concretely, let's analyze a situation in which $\vec{M} \parallel \hat{z}$ results from small current loops:

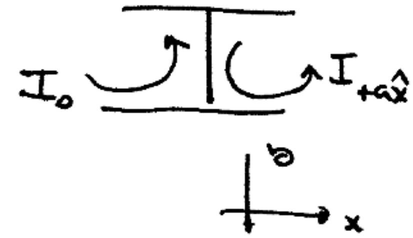


Let the squares have area a^2, and let the slice of material have thickness b. The magnetic moment in each cell is

$$\vec{m} = I a^2$$

where I is the current running around the cell. If the $\vec{m}$'s increase slightly from left to right, along the x direction

$$m^z(+a\hat{x}) - m^z(0)$$



$$= \underbrace{a^2}_{\text{area}} \cdot \left[I(+a\hat{x}) - I(0) \right]$$

$$= a^2 \left[-\text{net current flowing in boundary region} \right]$$

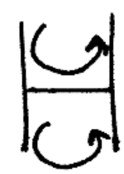
$$= a^2 \cdot a \cdot b \cdot (-j^y)$$

$$a \frac{\partial}{\partial x} m^z = -a^3 b\, j^y$$

replacing $M^z \cdot a^2 b = m^z$ $\qquad \frac{\partial}{\partial x} M^z = -j^y$

similarly, an M^z varying in the $\hat{y}$ direction gives

$$\frac{\partial}{\partial y} M^z = +j^x$$



this motivates

$$\vec{\nabla} \times \vec{M} = \vec{j}_b$$

To demonstrate this more generally, consider the $\vec{A}$ field due to a distribution of magnetic dipoles:

$$\vec{A}(x) = \int d^3y \, \frac{\mu_0}{4\pi} \, \vec{M}(y) \times \frac{\overrightarrow{x-y}}{(x-y)^3}$$

Now $\quad \dfrac{\overrightarrow{x-y}}{(x-y)^3} = \nabla_y \dfrac{1}{|\overrightarrow{x-y}|}$, so

$$\vec{A}(x) = \int d^3y \; \frac{\mu_0}{4\pi}\left(-\vec{\nabla}_y \frac{1}{|x-y|}\right) \times \vec{M}(y)$$

integrate by parts. Assuming $\vec{M}(y) \to 0$ at ∞,

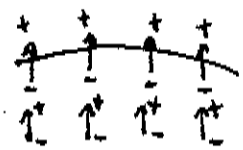
$$\vec{A}(x) = \int d^3y \; \frac{\mu_0}{4\pi} \frac{1}{|x-y|} \left(\vec{\nabla}\times\vec{M}(y)\right)$$

so $\vec{\nabla}\times\vec{M}$ is exactly the source of $\vec{A}$ equivalent to a macroscopic current.

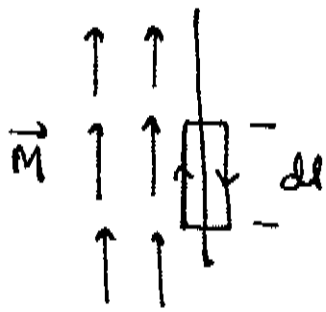
For polarizable media with a boundary, $\vec{P}$ changes discontinuously at the boundary. This gives rise to a surface bound charge:

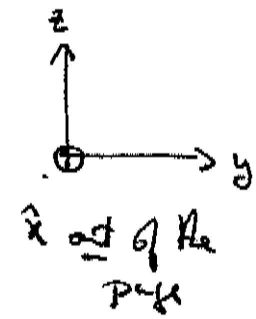
$$\rho_{bs} = \hat{n}\cdot\vec{P}$$

corresponding to the last charges sticking out at the boundary.



Similarly, in magnetic materials, $\vec{M}$ is discontinuous at a boundary, and this must be interpreted as a surface current. Consider a small loop integral at the surface: For $\vec{M} \parallel \hat{z}$ and a surface normal to $\hat{y}$:





$$\oint d\vec{l}\cdot\vec{M} = M^z\, dl$$

$$= -\underbrace{\mathcal{J}^x}_{\text{surface current density}} dl$$

for more general orientations

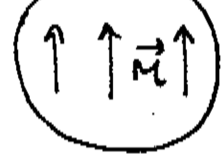
$$-\hat{n}\times\vec{M} = \vec{\mathcal{J}}$$

so, in all:

	Polarizable Media	Magnetic Media
macroscopic volume density	$\rho_b = -\vec{\nabla}\cdot\vec{P}$	$\vec{J}_b = \vec{\nabla}\times\vec{M}$
macroscopic surface density	$\rho_{bs} = \hat{n}\cdot\vec{P}$	$\vec{\mathcal{J}}_{bs} = -\hat{n}\times\vec{M}$

As we did in the electric case, let's compute the magnetic field due to a sphere with uniform dipole density. In the magnetic case, a uniform $\vec{M}$ gives zero volume current and a macroscopic surface current

$$\vec{J}_{b,s} = -\hat{n} \times \vec{M}$$



at polar angle θ, this is a current
($\vec{M} \parallel \hat{z}$)

$$\vec{J} = M \sin\theta \, \hat{\phi}$$

$\vec{J}$

We have already compute the magnetic fields due to a surface current density

$$\vec{J} = \omega \rho R \sin\theta \, \hat{\phi}$$

due to a rotating sphere of charge. For that case we found:

inside : <u>constant</u> $\vec{B} = \frac{2}{3} \mu_0 \omega \rho R \, \hat{z}$

outside : dipole field with: $\vec{m} = \frac{4\pi}{3} R^4 \omega \rho \, \hat{z}$

so, for a uniformly magnetized sphere:

inside: <u>constant</u> $\vec{B} = \frac{2}{3} \mu_0 |\vec{M}| \, \hat{z}$

outside: dipole field with $\vec{m} = \frac{4\pi}{3} R^3 |\vec{M}| \, \hat{z}$

compare this to the electrostatic case:

inside: constant $\vec{E} = -\frac{1}{3\epsilon_0} P \, \hat{z}$

outside: dipole field w. $\vec{p} = \frac{4\pi}{3} R^3 P \, \hat{z}$

Notice that the field inside the sphere is reversed from the electric to the magnetic case. This is a direct consequence of the requirement that $\vec{\nabla}\cdot\vec{B}=0$

uniform $\vec{P}$ uniform $\vec{M}$

Finally, as we did in the electrostatic case, we can define an auxiliary magnetostatic field which responds only to explicit ("free") current rather than the total current, which is a sum of free and bound current. Let

$$H = \frac{1}{\mu_0}B - M$$

then

$$\vec{\nabla}\times\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \vec{\nabla}\times\vec{M}$$

$$= \vec{J} - \vec{J}_b = \vec{J}_{free}$$

so the magnetostatic equations in the presence of magnetic media are:

$$\vec{\nabla}\times\vec{H} = \vec{J}_f \qquad \vec{\nabla}\cdot\vec{B} = 0$$

to be compared with the electrostatic equations

$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{D} = \rho_f \;, \quad D = \epsilon_0\vec{E}+\vec{P}$$

When we examined the electrostatic equations near a boundary between two media, we saw that they led to the boundary conditions

$$E_\parallel|_1 = E_\parallel|_2 \qquad D_\perp|_1 = D_\perp|_2$$

[and, if there is a boundary layer of ~~free~~ charge, there is a discontinuity in $D_\perp$.]

For magnetostatics, the same argument gives:

$$H_\parallel|_1 = H_\parallel|_2 \qquad B_\perp|_1 = B_\perp|_2$$

If there is a surface layer of free current, there is a discontinuity in $H_\parallel$.

These equations need to be supplemented by an equation relating $\vec{E}$ and $\vec{D}$ or $\vec{B}$ and $\vec{H}$. In the electrostatic case, we saw that it was reasonable that $\vec{P}$ is linearly related to $\vec{E}$:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

which implies that

$$\vec{D} = \epsilon_0(1+\chi_e)\vec{E} = \epsilon\vec{E}$$

The magnetostatic analogue of these equations are conventionally written:

$$\vec{M} = \chi_m \vec{H} \qquad \chi_m = \text{"magnetic susceptibility"}$$

so that

$$\vec{B} = \mu_0 (H + M) = \mu_0 (1 + \chi_m) \vec{H}$$

or

$$\vec{B} = \mu \vec{H}$$

where μ is the "permeability" of the material.

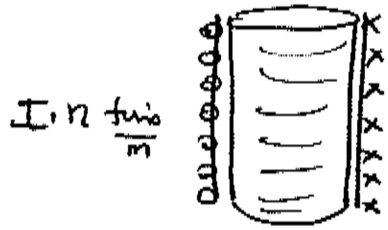
Please don't be confused by the form of this equation. $\vec{B}$ is the fundamental field which responds to all physical currents. $\vec{H}$ is only a construct useful for problem-solving. Nevertheless, there is a practical difference from electrostatics. In electrostatic problems, we typically impose a voltage difference between two points. This fixes

$$\Delta\phi = -\int d\vec{l} \cdot \vec{E}$$

For a dielectric inside a conductor, for example, we know $\vec{E}$ from the setup of the problem

$$E = V/d \qquad D = \epsilon V/d$$

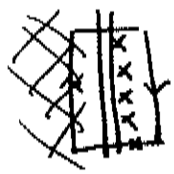
However, in magnetostatic problems, we typically work with a fixed current in a wire. This is a free current, and so it is H that responds to it directly. For a solenoid filled with a magnetic material:



The equat. $\vec{\nabla}\times\vec{H} = \vec{J}_f$ leads to the integral formula

$$\oint_{C=\partial S} d\vec{l}\cdot\vec{H} = I_{f,S}$$ modified Ampere's law

taken around a curve:



this implies

$$H_{inside} = I\cdot n$$

then

$$B_{inside} = \mu I n$$

If $\chi_m > 0$, this B is larger than the B expected if the solenoid is empty.