The Vector Potential

Nov. 20

In the previous lecture, we saw that the solution of the magnetostatic equations

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}$$

can be reduced to Laplace's equation in regions where $\vec{J}=0$. But, typically, $\vec{J}\neq 0$ and this approach is doomed to failure. Is there a more robust approach to these equations?

Let's begin with $\vec{\nabla}\cdot\vec{B}=0$. At the start of the course, I discussed the theorem that, if $\vec{\nabla}\cdot\vec{B}=0$ for a vector field $\vec{B}$ in a simply connected region R, then $\vec{B}$ can be represented in R as

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

Conversely, this representation ensures that $\vec{\nabla}\cdot\vec{B}=0$. $\vec{A}$ is called the <u>vector</u> <u>potential</u>.

If we put this expression for $\vec{B}$ into the second magnetostatic equation, we find

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\mu_0\vec{J}$$

$$+\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}=\mu_0\vec{J}$$

This has a piece that looks like three separate Laplace equations for the three components of $\vec{A}$.

It is not hard to find the general solution to these equations. Let's assume that we can find a solution with

$$\vec{\nabla}\cdot\vec{A} = 0$$

then we would need to solve

$$-\nabla^2\vec{A} = \mu_0 \vec{J}$$

This is exactly Laplace's equation for each component of $\vec{A}$, with source $\mu_0\vec{J}$. So

$$\vec{A}(\vec{x}) = \mu_0 \int d^3y \, \frac{1}{4\pi} \frac{1}{|x-y|} \vec{J}(y)$$

Now check $\vec{\nabla}\cdot\vec{A}$:

$$\vec{\nabla}\cdot\vec{A} = \mu_0 \int d^3y \, \frac{1}{4\pi} \nabla_x \frac{1}{|x-y|} \vec{J}(y)$$

$$= \mu_0 \int d^3y \, \frac{1}{4\pi} (-\vec{\nabla}_y) \frac{1}{|x-y|} \vec{J}(y)$$

integrate by parts, ignore surface term at $r=\infty$

$$= \mu_0 \int d^3y \, \frac{1}{4\pi} \frac{1}{|\vec{x}-\vec{y}|} \nabla_y\cdot\vec{J}(y)$$

Since in magnetostatics we are concerned with stationary currents, $\vec{\nabla}\cdot\vec{J} = 0$, and so $\vec{\nabla}\cdot\vec{A} = 0$. Thus, in general for magnetostatics:

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int d^3y \, \frac{1}{|x-y|} \vec{J}(y)$$

just to check:

$$\vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int d^3y \; \vec{\nabla}_x \frac{1}{|x-y|} \times \vec{J}(y)$$

$$= \frac{\mu_0}{4\pi}\int d^3y \left(-\frac{(\vec{x-y})}{(x-y)^3}\right)\times\vec{J}(y)$$

so

$$\vec{B} = \frac{\mu_0}{4\pi}\int d^3y \; \vec{J}(y)\times\frac{\vec{x-y}}{(x-y)^3}$$

and we see that the magnetostatic differential equations do imply the Biot-Savart law.

Is it odd that we must supplement the equation for $\vec{A}$ with a condition $\vec{\nabla}\cdot\vec{A}=0$. Actually, the equation

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

does not specify $\vec{A}$ unambiguously. For the electrostatic scalar potential, we know that the addition of a constant to ϕ does not show up in $\vec{E}$. For the vector potential, there is even more ambiguity:

if $$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

where λ is any scalar function,

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} \qquad \text{since} \quad \vec{\nabla}\times\vec{\nabla}\lambda = 0$$

The transformation from $\vec{A}$ to $\vec{A}'$, which does not affect any $\vec{B}$ fields or physical forces, is called a gauge transformation. (It is not so clear what this name signifies. I'll explain it later in the course.) Gauge transformations carry $\vec{A}(x)$ into an infinite family of other $\vec{A}(x)$ field configurations. The choice of one of these configurations is called "a choice of gauge" or "fixing the gauge".

The condition $\vec{\nabla}\cdot\vec{A}=0$ gives an almost unique specification of the gauge. Given $\vec{A}(\vec{x})$, let $\vec{A}'$ be a gauge-equivalent field configuration: $\vec{A}'=\vec{A}+\vec{\nabla}\lambda$. Then

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda$$

so if we insist that $\vec{\nabla}\cdot\vec{A}'=0$, we can satisfy this condition by choosing λ s.t.

$$-\nabla^2\lambda = \vec{\nabla}\cdot\vec{A}$$

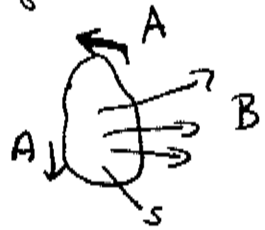
There is a unique solution for λ which $\to 0$ as $r\to\infty$. So, given $\vec{A}$, there is a gauge-equivalent $\vec{A}'$ satisfying $\vec{\nabla}\cdot\vec{A}'=0$. The condition

$$\vec{\nabla}\cdot\vec{A}=0$$

is called "specification of Coulomb gauge". We will find Coulomb gauge useful at other points in this course; other gauge choices are also useful in different situations.

A way to visualize $\vec{A}$ is via the equation

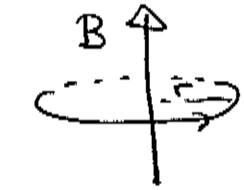
$$\int_{C=\partial S} d\vec{l}\cdot\vec{A} = \int_S d^2x\ \hat{n}\cdot\vec{B}$$



so, for example, we can represent a constant $\vec{B} = B\hat{z}$ field by choosing

$$\vec{A} = A(r)\,\hat{\phi}$$

with $A(r)$ chosen so that the circulation of A gives the required flux



$$\oint d\vec{l}\cdot\vec{A} = 2\pi r\,A(r) = \int d^2x\,\hat{n}\cdot\vec{B} = \pi r^2 B$$

so

$$\vec{A} = \tfrac{1}{2} r B\,\hat{\phi}$$

or, in general

$$\vec{A} = -\tfrac{1}{2}\,\vec{r}\times\vec{B}$$

for $\vec{B} = B\hat{z}$ $$\vec{A} = \left(-\tfrac{1}{2}yB\,,\ \tfrac{1}{2}xB\,,\ 0\right)$$

Another representation of a constant B field is

$$\vec{A}' = (-yB\,,\ 0\,,\ 0)$$

Indeed the difference is a gradient, as required:

$$\vec{A}'-\vec{A} = \left(-\tfrac{y}{2}B\,,\ -\tfrac{x}{2}B\,,\ 0\right) = -\vec{\nabla}\left(xy\tfrac{B}{2}\right)$$

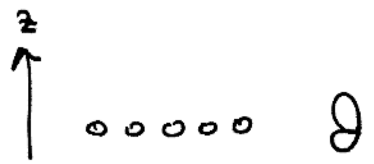
Let's solve some problems using $\vec{A}$. First of all, consider the $\vec{B}$ field created by a sheet of current

$$\vec{J} = J\,\hat{x}\,\delta(z)$$  $J = C/\mathrm{msec}$

We need to solve the equations.

$$\vec{\nabla}\cdot\vec{A} = 0 \qquad\qquad -\nabla^2\vec{A} = \mu_0 \vec{J}$$

The sheet of current is analogous to a sheet of charge as a source of A^x.



$\Longleftrightarrow$

$$\phi = -\frac{1}{2\epsilon_0}|z|\,\rho$$

$$A^x = -\frac{\mu_0}{2}|z|\,\mathcal{J} \qquad \Longleftrightarrow \qquad E^z = \frac{\rho}{2\epsilon_0}$$

for $z > 0$

$$\vec{\nabla}\times\vec{A} = \frac{\partial}{\partial z}A^x\,\hat{y} = -\frac{\mu_0 \mathcal{J}}{2}\,\hat{y}$$

for $z < 0$

$$\vec{\nabla}\times\vec{A} = \frac{\partial}{\partial z}A^x\,\hat{y} = +\frac{\mu_0 \mathcal{J}}{2}\,\hat{y}$$

which is exactly what we find in the previous lecture.

We can also present another solution to the solenoid problem using $\vec{A}$. The solenoid corresponds to a surface current

$$\vec{\mathcal{J}} = In\,\hat{\phi} \qquad\qquad n = \text{turns/m}$$

$$= (-In\sin\phi,\ In\cos\phi,\ 0)$$

This is the source for an $\vec{A}$ field. The whole problem is independent of $\hat{z}$. We need to solve

$$-\nabla^2 \vec{A} = 0$$

in the interior and in the exterior of the cylinder, with a surface source $\vec{J}$ on the cylinder. Let's first find A^x. The problem of solving for A^x is equivalent to solving for an electrostatic potential ϕ with a surface charge

$$\rho = -I_n \sin\phi.$$

So use methods from electrostatics. We are working in cylindrical coordinates and are looking for a solution independent of z.

The most general interior solution, regular at $r=0$, is

$$\phi_{in} = \sum_{m=-\infty}^{\infty} A_m r^{|m|} e^{im\phi}$$

The most general exterior solution, $\phi \to$ const at $r=\infty$, is

$$\phi_{out} = \sum_{m=-\infty}^{\infty} B_m r^{-|m|} e^{im\phi}$$

In the analogy to electrostatics, the tangential E field derived from ϕ should be continuous across the boundary, while the normal component of E should have a discontinuity.

$$\text{"}E_{\parallel}\text{"}|_{in} = -\frac{1}{r}\frac{\partial}{\partial\phi}\phi_{in} = \text{"}E_{\parallel}\text{"}|_{out} \quad \text{at } r=R$$

$$\text{"}E_{\perp}\text{"}|_{out} - \text{"}E_{\perp}\text{"}|_{in} = \left(-\frac{\partial}{\partial r}\phi_{out}\right) - \left(-\frac{\partial}{\partial r}\phi_{in}\right) = \mu_0 \rho$$

replaces $\frac{1}{\epsilon_0}$

The $E_\parallel$ condition implies, since the functions $e^{im\phi}$ are orthogonal

$$A_m R^{|m|} = B_m R^{-|m|}$$

The $E_\perp$ condition is

$$\sum_{m=-\infty}^{\infty} |m| \left(B_m R^{-|m|-1} + A_m R^{|m|-1}\right) e^{im\phi}$$

$$= -\mu_0 I n \sin\phi$$

Now, $\sin\phi = \frac{1}{2i}\left(e^{i\phi} - e^{-i\phi}\right)$, so all B_m, A_m can be set to zero except for $m = \pm 1$. Then

$$\phi_{in} = -\mu_0 \frac{In}{2} \frac{r}{R} \sin\phi \qquad \phi_{out} = -\mu_0 \frac{In}{2} \frac{R^2}{r} \sin\phi$$

gives the solution. Solving similarly for A's, we have, instead

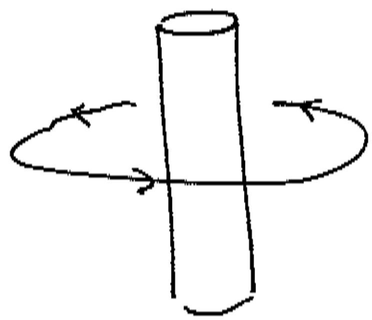
$$\vec{A}_{in} = +\frac{\mu_0 I n}{2} \frac{r}{R} \left(-\sin\phi, \cos\phi, 0\right)$$

$$= \frac{1}{2} r \left(\mu_0 I n\right) \hat{\phi}$$

This is the $\vec{A}$ field of a constant B field $\vec{B} = \mu_0 I n \hat{z}$.

Notice that $\vec{A}$ is nonzero outside:

$$\vec{A}_{out} = \frac{\mu_0 I n}{2} \frac{R^2}{r} \hat{\phi}$$



But this is exactly what we need so that, if C is a circle lying outside the cylinder: $\oint_C dl \cdot A = 2\pi r A_\phi = \mu_0 I n \pi R^2$

which is exactly the flux of $\vec{B}$ thru the circle.
It is not hard to show explicitly that

$$\vec{\nabla} \times \vec{A}_{out} = 0$$

so there is no $\vec{B}$ field outside the cylinder, as required.

In electrostatics, we used the formula

$$\phi(x) = \int d^3y \, \frac{1}{4\pi\epsilon_0 |\vec{x}-\vec{y}|} \rho(\vec{y})$$

as the basis for making a general expansion of electrostatic fields far from a charge distribution — the multipole expansion. We can use our expression for $\vec{A}(x)$ to carry out a multipole expansion in magnetostatics: Start from

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int d^3y \, \frac{1}{|x-y|} \vec{J}(y)$$

assume that there is an R s.t. $|\vec{J}(y)| = 0$ for $|\vec{y}| > R$, and expand, for $|\vec{x}| > R$:

$$\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{x} + \frac{\hat{x}}{x^2} \cdot \vec{y} + \frac{1}{x^3}\left[\frac{3}{2}(\hat{x}\cdot\vec{y})^2 - \frac{1}{2} y^2\right]$$

$$+ \cdots$$

The corresponding terms in $\vec{A}$ are the magnetic monopole, dipole, quadrupole, etc. fields the monopole term is

$$\vec{A}(\vec{x})\Big|_{\text{monopole}} = \frac{\mu_0}{4\pi} \int d^3y \; \frac{1}{x} \; \vec{J}(y)$$

$$= \frac{\mu_0}{4\pi} \frac{1}{x} \left[\int d^3y \; \vec{J}(y) \right]$$

this contribution would lead to a $1/x^2$ B field. However, it always vanishes! To see this, note that, if $f(\vec{y})$ is any well-behaved function,

$$\int d^3y \; \vec{J}(y) \cdot \vec{\nabla} f(y) = 0$$

since this integral equals.

$$\int d^3y \; \vec{\nabla} \cdot \left(f(y) \vec{J}(y) \right) = \int d^2y \; \hat{n} \left(f \vec{J}(y) \right) = 0$$

using $\vec{\nabla} \vec{J} = 0$ in the first step and $\vec{J} = 0$ for $|y| > R$ in the last.
If we choose $f(\vec{y}) = y^i$ $\vec{\nabla}^k f = \delta^{ik}$, we see that

$$\int d^3y \; \vec{J} = 0$$

for any stationary current. If $\vec{J}$ is the current in a wire, the proof is easier. Taking the limit in which $\vec{J}$ is concentrated on a wire:

$$\vec{A} = \frac{\mu_0}{4\pi} I \oint d\vec{l} \; \frac{1}{|x-y|}$$

The monopole term in this expression is:

$$\vec{A}\Big|_{mon.} = \frac{\mu_0}{4\pi}\frac{1}{x}\, I \oint d\vec{l}$$

but $\oint d\vec{l} = 0$ for a closed loop.

Thus, magnetostatic fields due to currents fall off as $\frac{1}{x^3}$ or faster: the first nonzero term in their multipole expansion is the dipole. If there exist sources of magnetic monopole fields, those lie outside our discussion of magnetostatics up to now.

Anyway, now turn to the dipole term:

$$\vec{A}_{dipole}(x) = \frac{\mu_0}{4\pi}\int d^3y \left(\frac{\hat{x}}{x^2}\cdot\vec{y}\right)\vec{j}(y)$$

To aid the next manipulation, write this in components as

$$\vec{A}^{k}_{dipole}(x) = \frac{\mu_0}{4\pi}\frac{x^i}{x^3}\int d^3y\; y^i\, j^k(y)$$

Now go back to the mathematical result on p. 10 and insert $f = y^i y^k$

$$0 = \int d^3y\; \vec{j}\cdot\vec{\nabla}(y^i y^k) = \int d^3y\,(j^i y^k + j^k y^i)$$

so the above

$$= \frac{\mu_0}{8\pi}\frac{x^i}{x^3}\int d^3y\;(y^i j^k - y^k j^i)$$

$$= -\frac{\mu_0}{8\pi}\frac{1}{x^3}\int d^3y\;\left[\vec{x}\times(\vec{y}\times\vec{j})\right]^k$$

Then

$$\vec{A}_{dip}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{x^3}$$

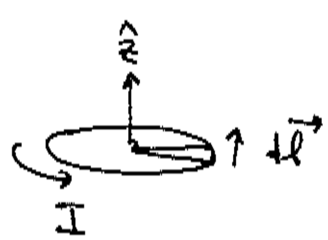
where

$$\vec{m} = \frac{1}{2} \int d^3y \, (\vec{y} \times \vec{J})$$

is the magnetic dipole moment. For a current loop

$$\vec{m} = -\frac{1}{2} I \int d\vec{l} \times \vec{y}$$

For a loop in the x,y plane



$$\vec{m} = +\frac{1}{2} I \; 2\pi R \cdot R \, \hat{z}$$

$$= I \, \pi R^2 \, \hat{z}$$

In fact, in general,

$$\frac{1}{2} \hat{y} \times d\vec{l} = da \, \hat{n}$$

where da is the area of the indicated triangle and $\hat{n}$ is the normal to the surface. So

$$\vec{m} = I \int da \, \hat{n}$$

This dipole moment is exactly the one we found when we analyzed the asymptotic field of a current loop. Now let's compute the corresponding B field:

$$(\vec{\nabla}\times\vec{A}_{dip.})^k = \epsilon^{ijk}\,\nabla^i A^j_{dip}$$

$$= \epsilon^{ijk}\,\nabla^i\,\epsilon^{jlm}\,\frac{\mu_0}{4\pi}\,\frac{m^l x^m}{x^3}$$

$$= \epsilon^{ijk}\epsilon^{jlm}\,\frac{\mu_0}{4\pi}\,m^l\left(\frac{\delta^{im}x^2 - 3x^m x^i}{x^5}\right)$$

$$= (-\delta^{il}\delta^{km} + \delta^{im}\delta^{kl})\,\frac{\mu_0}{4\pi}\,m^l\left(\frac{\delta^{im}x^2 - 3x^m x^i}{x^5}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{1}{x^5}\Big[-m^k x^2 + m^k x^2\,\delta^{im}\delta^{im}$$

$$\delta^{im}\underline{\delta^{im} = 3} \qquad + 3x^k\,\vec{x}\cdot\vec{m} - 3m^k x^2\Big]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{x^5}\left[3x^k\,\vec{x}\cdot\vec{m} - m^k x^2\right]$$

so

$$\vec{B}_{dipl.} = \vec{\nabla}\times\vec{A}_{dip} = \frac{\mu_0}{4\pi}\left(\frac{3\vec{x}\;\vec{x}\cdot\vec{m} - x^2\vec{m}}{x^5}\right)$$

which is indeed exactly of the form of a dipole field, and is also in precise agreement with our analysis of a current loop.