Computation of Magnetic Fields

In the previous lecture, we derived the magnetostatic field equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

the first of these implies $\int_S d^2x \, \hat{n}\cdot\vec{B} = 0$ for any surface whatever. The second is equivalent to Ampere's law:

$$\oint_{C=\partial S} d\vec{l}\cdot\vec{B} = \mu_0 I_S$$

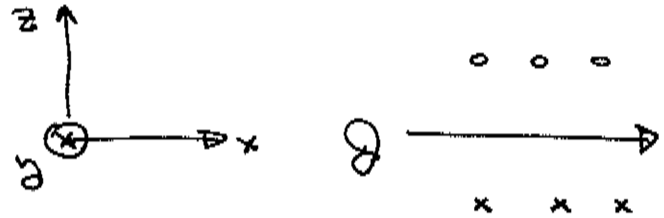
In this lecture, I will use these equations to compute magnetic field configurations.

First of all, there are current configurations of special symmetry for which we can deduce the magnetic fields only from symmetry and Ampere's law. Here are some examples.

Consider a flat sheet of conductor carrying a surface current $\vec{\mathcal{J}}$ C/m·sec

(i.e. a strip of width b along the current flow carries a current $I = b\mathcal{J}$.)

If we let the surface be the x,y plane and we let the current run along $\hat{x}$, the problem is solved by a magnetic field



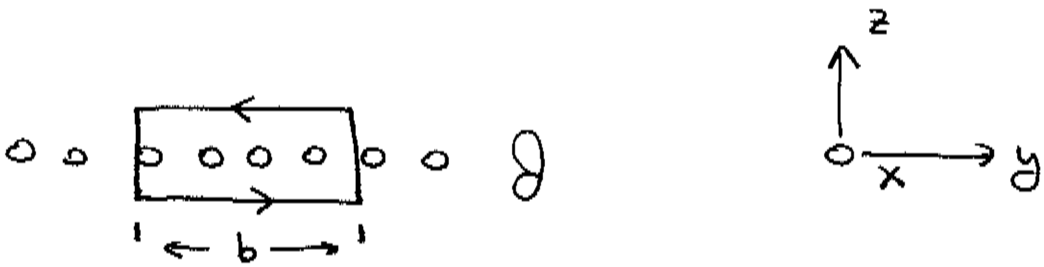
$-B\hat{y}$ $\quad z>0$

$+B\hat{y}$ $\quad z<0$

Notation o = vector coming out of the paper

x = vector going into the paper

To determine the strength of B, use Ampere's law with a contour



$$\oint d\vec{l}\cdot\vec{B} = 2B(z)\cdot b = \mu_0 I = \mu_0 b \mathcal{J}$$

so

$$B = \frac{\mu_0 \mathcal{J}}{2}$$ independent of height above the surface.

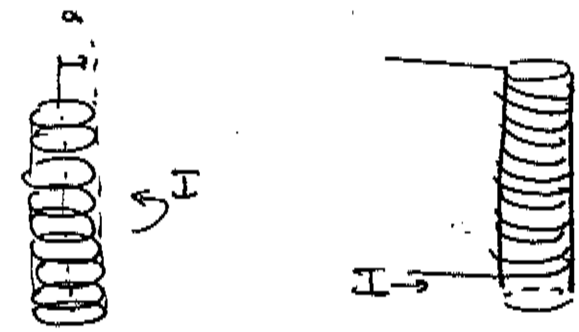
again

$$\vec{B} = \begin{cases} -\frac{\mu_0 \mathcal{J}}{2}\hat{y} & z>0 \\ +\frac{\mu_0 \mathcal{J}}{2}\hat{y} & z<0 \end{cases}$$

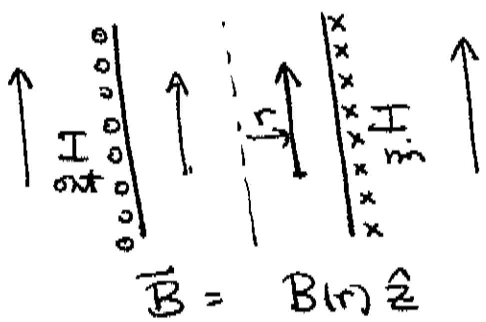
This field satisfies $\vec{\nabla}\times\vec{B}=0$ above and below the surface and $\vec{\nabla}\cdot\vec{B}=0$ everywhere.

The surface current is the magnetostatic analogue of the surface charge in electrostatics

Next, consider a "solenoid", idealized as a stack of current loops, densely packed with $\underline{n}$ loops/m. Each loop has radius a and carries current I.

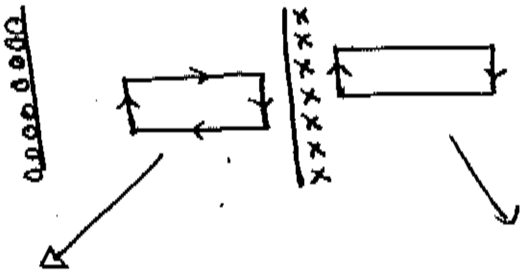


One can make a good approximation to a solenoid by wrapping a single wire tightly around a cylinder of radius a. Make the idealization that the solenoid extends infinitely in the $\hat{z}$ direction. Then we can find a solution with $\vec{B}\parallel\hat{z}$.



$$\vec{B} = B(r)\,\hat{z}$$

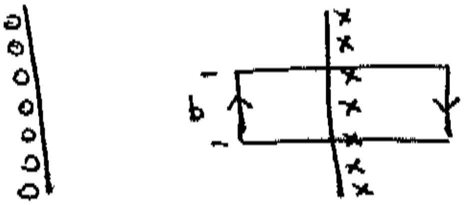
To find B(r), use Ampere's law on loops:



$B(r_1) = B(r_2)$ if r_1, r_2 are both inside

$B(r_1) = B(r_2)$ if r_1, r_2 are both outside

and, in particular, if $B \to 0$ as $r \to \infty$, $\vec{B} = 0$ outside.



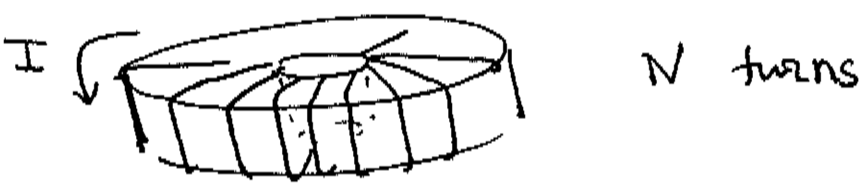
$$\oint dl \cdot B = B_{inside} \cdot b = \mu_0 I n b$$

so, $\vec{B}$ is constant in each region!

$$\vec{B} = \begin{cases} \mu_0 I n \, \hat{z} & \text{inside} \\ 0 & \text{outside.} \end{cases}$$

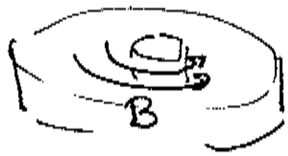
The solenoid is the analogue in magnetostatics of the parallel-plate capacitor.

Finally, consider a wire carrying current I wrapped tightly on a surface of rotation about the $\hat{z}$ axis, with N turns around the circle

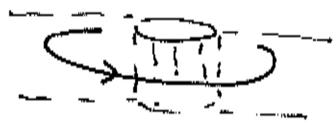


We might guess that, in this situation

$$\vec{B} = B(r,z)\,\hat{\phi}$$



Using Ampere's law on a circle



$$\oint d\vec{l}\cdot\vec{B} = 2\pi r\, B(r,z) = \begin{cases} \mu_0 I N & \text{if the loop is inside} \\ 0 & \text{if the loop is outside} \end{cases}$$

whatever the shape of the container.

So

$$\vec{B} = \begin{cases} \frac{\mu_0 I N}{2\pi r}\,\hat{\phi} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

It is instructive to show explicitly that this satisfies $\vec{\nabla}\times\vec{B} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$ inside the container. In cylindrical coordinates

$$\vec{\nabla}\cdot\vec{B} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\vec{B}$$

$$= \hat{r}\cdot\frac{\partial}{\partial r}\left[\left(\frac{\mu_0 I N}{2\pi}\frac{1}{r}\right)\hat{\phi}\right] + \hat{\phi}\,\frac{1}{r}\,\frac{\mu_0 I N}{2\pi r}\left(\frac{\partial}{\partial\phi}\hat{\phi}\right)$$

$$+ \text{(all other terms are obviously zero)}$$

Now, from the explicit expression for $\hat{r}$, $\hat{\phi}$:

$$\frac{\partial}{\partial r}\hat{r} = 0 \quad \frac{\partial}{\partial r}\hat{\phi} = 0 \qquad \frac{\partial}{\partial \phi}\hat{r} = \hat{\phi} \qquad \frac{\partial}{\partial \phi}\hat{\phi} = -\hat{r}$$

Then both terms written explicitly on p. 5 are zero, $\vec{\nabla}\cdot\vec{B} = 0$

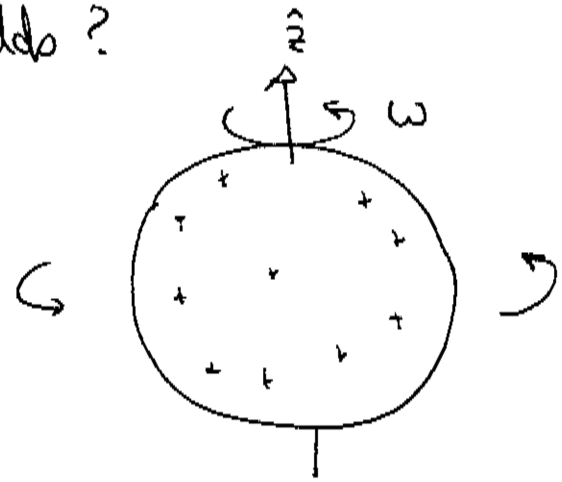
$$\vec{\nabla}\times\vec{B} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right)\times\left(\frac{\mu_0 I N}{2\pi}\frac{1}{r}\hat{\phi}\right)$$

$$= (\hat{r}\times\hat{\phi})\frac{\mu_0 I N}{2\pi}\left(\frac{\partial}{\partial r}\frac{1}{r}\right) + \hat{\phi}\times\frac{\partial\hat{\phi}}{\partial\phi}\frac{1}{r}\frac{\mu_0 I N}{2\pi}\frac{1}{r}$$

$+$ (all other terms $= 0$)

$$= \hat{z}\left(-\frac{\mu_0 I N}{2\pi}\frac{1}{r^2}\right) + \hat{\phi}\times(-\hat{r})\frac{\mu_0 I N}{2\pi r^2} = 0 \;!$$

When symmetry is not enough to solve the problem, we need to use the methods we have developed in electrostatics for analyzing partial differential equations. One strategy is to define a <u>magnetic</u> <u>scalar</u> <u>potential</u>. Consider the following problem: A sphere with charge ρ C/m² on its surface rotates about the $\hat{z}$ axis at angular velocity ω. What are the magnetic fields?



The current lies entirely on the surface of the sphere. Inside or outside, $\vec{J} = 0$. Then, separately inside and outside, $\vec{B}$ satisfies:

$$\vec{\nabla} \times \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

This means that we can represent $\vec{B} = \vec{\nabla} \phi_B$, where ϕ_B solves the Laplace equation: $-\nabla^2 \phi_B = 0$.

Inside the sphere, $\vec{B}$ is regular as $r \to 0$. So the most general solution of the Laplace equation is

$$\phi_{B,in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Outside, $\vec{B} \to 0$ as $r \to \infty$. So the most general solution of the Laplace equation is

$$\phi_{B,out} = \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

The two solutions must knit together correctly on the boundary. Let's compute the boundary values $(r = R)$

$$B_{\perp,in} = -\frac{\partial}{\partial r}\phi_{B,i} = -\sum_{l=0}^{\infty} A_l \cdot l\, r^{l-1} P_l(\cos\theta)$$

$$B_{\perp,out} = -\frac{\partial}{\partial r}\phi_{B,out} = +\sum_{l=0}^{\infty} B_l (l+1) \frac{1}{r^{l+2}} P_l(\cos\theta)$$

$$B_{\parallel,in} = -\frac{1}{r}\frac{\partial}{\partial\theta}\Phi_{B\,in} = -\sum_{\ell=0}^{\infty} A_\ell\, r^{\ell-1}\frac{\partial}{\partial\theta}P_\ell(\cos\theta)$$

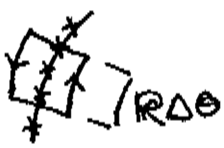
$$B_{\parallel,out} = -\frac{1}{r}\frac{\partial}{\partial\theta}\Phi_{B,out} = -\sum_{\ell=0}^{\infty} B_\ell\,\frac{1}{r^{\ell+2}}\frac{\partial}{\partial\theta}P_\ell(\cos\theta)$$

To get a condition for $B_\perp$, integrate $\vec{\nabla}\cdot\vec{B}=0$ over a small box poking through the surface:

$$\int d^2x\,\hat{n}\cdot\vec{B}$$
$$= (\text{Area})\,(B_{\perp\,out} - B_{\perp\,in})$$

since $\vec{\nabla}\cdot\vec{B}=0$, this must vanish, so $B_{\perp\,out} = B_{\perp\,in}$

To get a condition for $B_\parallel$, apply Ampere's law to the line integral:



$$\oint d\ell\cdot B = R\Delta\theta\,(B_{\theta,out} - B_{\theta\,in})$$
$$= \mu_0 I \quad = \mu_0\rho\cdot R\Delta\theta\cdot v(\theta)$$

where the velocity of the charges at θ is $(\omega R\sin\theta)$

$$B_{\theta,out} - B_{\theta,in} = \mu_0\,\rho\cdot\omega R\sin\theta$$
$$= \mu_0\,\rho\,\omega\,R\left[-\frac{\partial}{\partial\theta}P_1(\cos\theta)\right]$$

so we can consistently set all $A_l, B_l = 0$ for $l \neq 1$.

For $l=1$

$$B_\perp: \quad -A_1 = B_1 \frac{2}{R^3}$$

$$B_\parallel: \quad \left(-B_1 \frac{1}{R^3}\right) - (-A_1) = -\mu_0 \omega R \rho$$

The solution is:

$$A_1 = -\frac{2\mu_0 \omega \rho R}{3} \qquad B_1 = \frac{\mu_0 \omega \rho R^4}{3}$$

so inside

$$\Phi_{B,in} = -\frac{2\mu_0 \omega \rho R}{3} r\cos\theta = -\frac{2\mu_0 \omega \rho R}{3} z$$

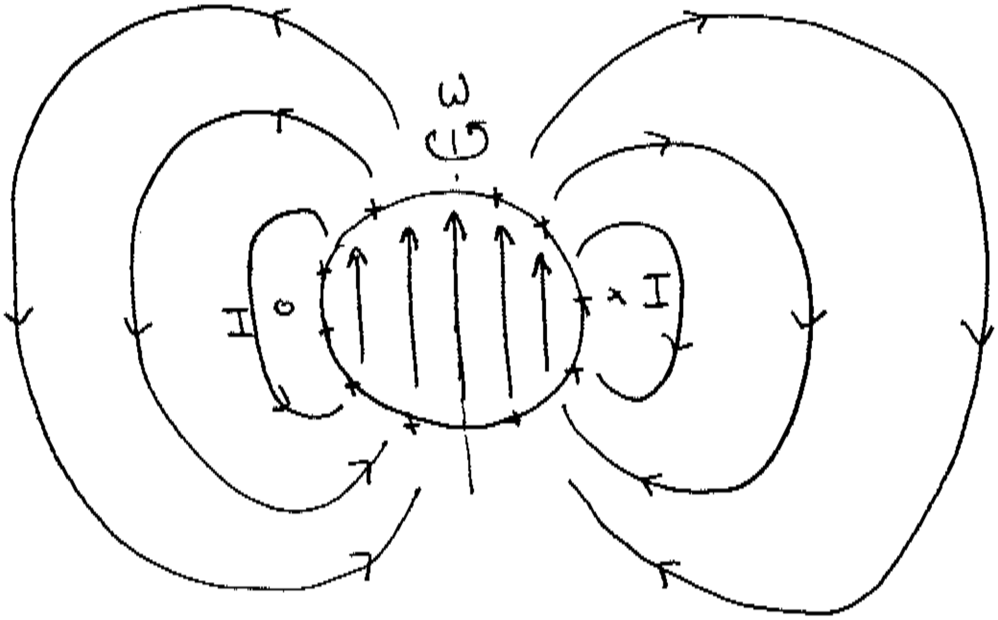
$$\vec{B} = +\frac{2\mu_0 \omega \rho R}{3} \hat{z}$$

outside:

$$\Phi_{B,out} = \frac{\mu_0 \omega \rho R^4}{3r^2} \cos\theta$$

$$= \frac{\mu_0}{4\pi} \frac{\hat{r}}{r^2} \cdot \vec{P}$$

where $\vec{P} = \frac{4\pi}{3} R^4 \omega \rho \, \hat{z}$



Notice that the $\vec{B}$ field is sourceless, as required by the field equation $\vec{\nabla}\cdot\vec{B} = 0$.