Magnetism – 2

Nov. 15

It will be good to write the Biot-Savart law in a form ~~that~~ we can manipulate more easily. First, however, I would like to briefly discuss particle motion in magnetic fields.

In the last lecture, we wrote the magnetic force as

$$\vec{F} = Q\,(\vec{v} \times \vec{B})$$

This would be a good time to mention that the units of $\vec{B}$ are

$$N/(C\,m/sec) = N/A\cdot m$$

The standard unit of B is the tesla:

$$1 \text{ Tesla} = 1\ N/A\cdot m$$

Another common unit is the gauss:

$$1 \text{ gauss} = 10^{-4}\ T$$

The earth's magnetic field is about $\frac{1}{2}$ g ; 1 T is a very strong laboratory field. (The magnets of the LHC, which represent the state of the art in high-field large-scale magnetic systems, will operate at 7 T.)

To get an understanding of the force law, consider particle motion in a constant B field $\vec{B} = B\hat{z}$.

(I will discuss how to produce a constant B field later in the lecture.) The Newton equation of motion is

$$m\dot{\vec{V}} = Q\,(\vec{V}\times\vec{B})$$

since the force is perpendicular to $\vec{B}$, $\dot{V}^z = 0$, $V^z = (\text{const})$

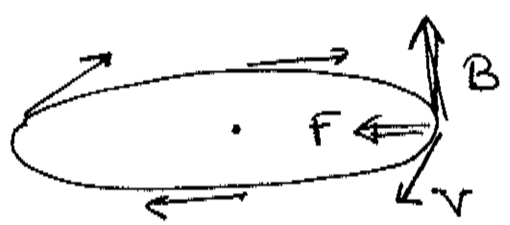
The x and y components of V satisfy:

$$m\dot{V}^x = Q\,V^y B \qquad\qquad m\dot{V}^y = -Q\,V^x B$$

A solution to this equation with two free initial conditions is:

$$V^x = A\cos(\omega_c t + \phi_0)$$
$$V^y = -A\sin(\omega_c t + \phi_0)$$

with $\omega_c = \dfrac{QB}{m}$; this frequency is called the cyclotron frequency. The x,y projection of $\vec{V}$ orbits in the counterclockwise direction, as we can see from the force diagram



Integrating with respect to t, we find

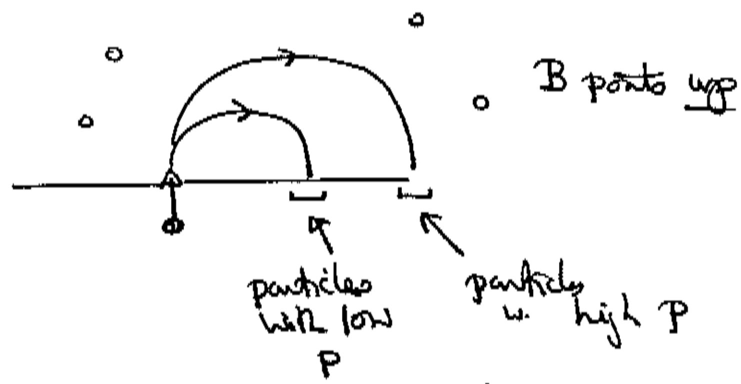
$$\vec{X}(t) - \underbrace{\vec{X}_0}_{} = \left(B(\sin\omega_c t + \phi_0),\ B[(\cos\omega_c t + \phi_0) - 1],\ Ct\right)$$

with $B = A/\omega_c$

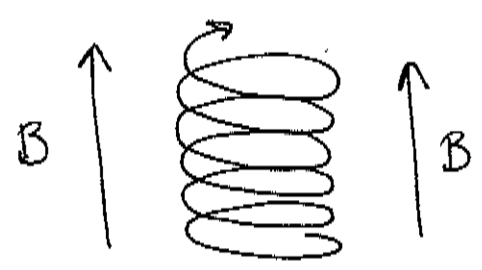
If $C = 0$ and there is no motion in $\hat{z}$, the particle moves counterclockwise in a circle of radius:

$$R = \frac{v}{\omega_c} = \frac{mv}{QB} = \frac{P}{QB}$$

The higher the momentum, the larger the radius. This is the principle of the magnetic spectrometer:



If C is not equal to zero, the path is not a circle but rather a spiral (better: a helix).



For low p and large B, the particle moves parallel or antiparallel to the field lines while making a tight circle normal to the field lines.

You know that it is problemmatic to discuss energy conservation with velocity-dependent forces. A typical damping force for an oscillator is

$$\vec{F} = -\gamma \vec{v}$$

and this is constructed to violate energy conservation. However magnetic forces are an exception. The work done by a magnetic field is

$$\dot{W} = \vec{v} \cdot \vec{F} = \vec{v} \cdot Q(\vec{v} \times \vec{B}) = 0$$

so, magnetic fields do no work and cannot change the energy of a particle.

Now let's go back to our consideration of the Biot-Savart law. For a volume current $\vec{J}$, this law reads.

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3y \; \vec{J}(y) \times \frac{\vec{r}}{r^3}$$

where $\vec{r} = (\vec{x-y})$. We can write this as

$$= \frac{\mu_0}{4\pi} \int d^3y \; \vec{J}(y) \times \left(-\vec{\nabla}_x \frac{1}{|x-y|}\right)$$

Now compute the action of $\vec{\nabla}$ on $\vec{B}$, just as we did for our integral expression for the electrostatic field. First,

$$\vec{\nabla} \cdot \vec{B} = -\frac{\mu_0}{4\pi} \int d^3y \; \vec{\nabla}_x \cdot \left(\vec{J}(y) \times \vec{\nabla}_x \frac{1}{|x-y|}\right)$$

In components, the right-hand side is

$$-\frac{\mu_0}{4\pi}\int d^3y \quad \epsilon^{ijk} \; j^j(y) \; \underbrace{\nabla_x^i \nabla_x^k \frac{1}{|\vec{x-y}|}}$$

which vanishes, because this expression ↑ is symmetric under $i \leftrightarrow k$.

In a similar way, we can compute the curl of B:

$$\vec{\nabla}\times\vec{B} = -\frac{\mu_0}{4\pi}\int d^3y \quad \vec{\nabla}_x \times (\vec{j}(\vec{y}) \times \vec{\nabla}_x)\frac{1}{|x-y|}$$

$$= -\frac{\mu_0}{4\pi}\int d^3y \left(\vec{j}(\vec{y}) \; \vec{\nabla}_x\cdot\vec{\nabla}_x \frac{1}{|x-y|} - (\vec{j}\cdot\vec{\nabla}_x)\vec{\nabla}_x \frac{1}{|x-y|}\right)$$

The second term can be shown to be zero

$$\int d^3y \; \vec{j}(y)\cdot\vec{\nabla}_x \left(\vec{\nabla}\frac{1}{|x-y|}\right) = -\int d^3y \; \vec{j}(y) \vec{\nabla}_y \left(\vec{\nabla}_x \frac{1}{|x-y|}\right)$$

integrate by parts and assume $j \to 0$ as $r \to \infty$ $\quad = \int d^3y \, (\vec{\nabla}_y\cdot\vec{j}) \left(\nabla_x \frac{1}{|x-y|}\right)$

$$= 0$$

since for steady currents $\vec{\nabla}\cdot\vec{j} = 0$

In the first term we have the Laplacian applied to a $\frac{1}{r}$ potential.

$$-\nabla_x^2 \frac{1}{|x-y|} = 4\pi\,\delta^{(3)}(\vec{x-y})$$

so

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{j}$$

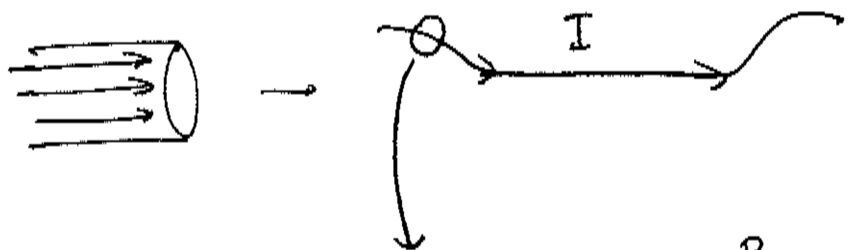
You might be worried that the whole contribution comes from the singularity where $\vec{x}=\vec{y}$. To check this, let's study an integral form of this equation: Integrate over a surface S:

$$\int_S d^2x\,\hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \int_{C=\partial S} d\vec{l}\cdot\vec{B} = \int d^2x\,\hat{n}\cdot\mu_0\vec{J}$$

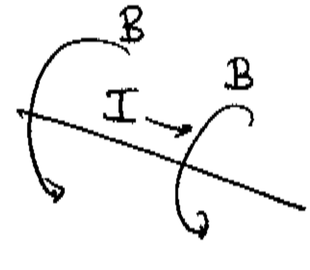
or

$$\int_{C=\partial S} d\vec{l}\cdot\vec{B} = \mu_0(\underline{\text{flux}}\text{ of }\vec{J}\text{ through }S)$$

This is <u>Ampere's law</u>. To check the form of the singular terms, we might as well consider $\vec{J}$ as concentrated on a wire:



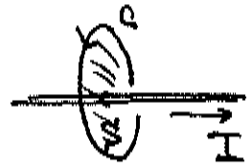
Very close to the wire, we can approximate that it is long and straight:



we compute the B field in this case and found

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{d} \hat{\phi}$$

at a distance d from the wire. Then



$$\oint_C d\vec{l}\,\vec{B} = 2\pi d \cdot B_\phi = \mu_0 I$$

and I is indeed the flux of $\vec{J}$ through the surface.
Note also that this $\vec{B}$ field is manifestly sourceless:

$\vec{\nabla} \cdot \vec{B} = 0$.

We have shown, then, that the Biot-Savart law leads to the magnetostatic field equations

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

and that the latter is equivalent to Ampere's law:

$$\int_{C=\partial S} d\vec{l} \cdot \vec{B} = \mu_0 I_S$$

where I_S is the current (C/sec) flowing through S.