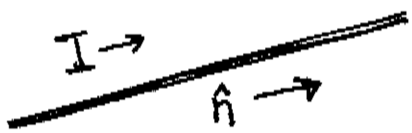
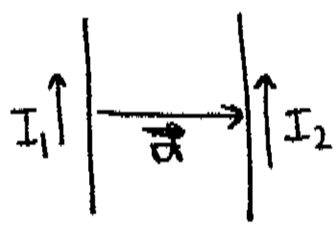
Magnetism

Up to this point in the course, we have discussed only the electrical forces between static charges. You might think that these forces generalize naturally when charges are in motion. You would be wrong. There is a new story waiting for us there.

As a first example, consider the interaction of two long straight wires. Describe a straight wire as a line carrying current I in the direction $\hat{n}$



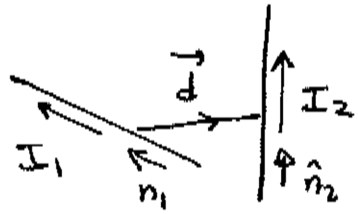
A wire is electrically neutral, so it should lead to no electrostatic forces. Nevertheless, two parallel wires exert a force on one another



$$\vec{F}_{2\leftarrow 1} = (\text{const}) \cdot I_1 I_2 \left(-\frac{\hat{d}}{d}\right)$$

if $\vec{d}$ is the $\perp$ between the two wires

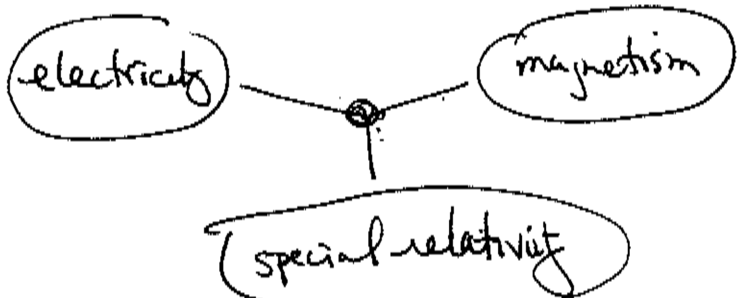
For two long wires at an angle



$$\vec{F}_{2\to1} \propto (\text{const})\ I_1 I_2\ \hat{n}_1\cdot\hat{n}_2\,(-\hat{d})$$

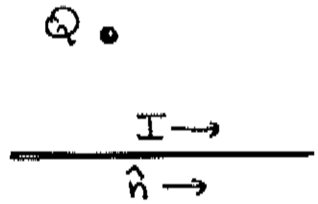
This force would seem to have nothing to do with the electrostatic interaction, and, in fact, it has another name — magnetism. Whereas the electrostatic force is simple and easy to understand, magnetism is more complex and much harder to visualize. For the moment, we will discuss magnetic interactions as a completely different force of Nature.

However, next term we will see that electrostatics and magnetism are inextricably linked through the theory of relativity. In fact, we will see that in the triad:

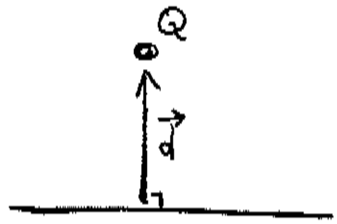


any two branches require the third. This is the most amazing single aspect of classical physics!

For the moment, though, I would like to develop the mathematical description of magnetism in its own right. To get a clearer picture of the magnetic force, let's consider the force on a point charge due to a wire:



If Q is stationary, there is no force, in accord with electrostatics. However, if Q is in motion with velocity $\vec{v}$ (in m/sec), it experiences a force



$$\vec{F} = (\text{const})\ Q \cdot I \cdot \frac{1}{d} \cdot \left(-\hat{d}\ \vec{v}\cdot\hat{n} + \hat{n}\ \vec{v}\cdot\hat{d}\right)$$

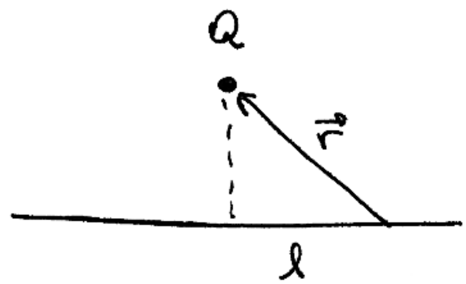
where $\vec{d}$ is the perpendicular from the wire to the charge. This force is

- proportional to Q
- proportional to I
- perpendicular to $\vec{v}$ $\qquad \vec{v}\cdot\vec{F} = 0$
- decreasing as $1/d$

We can try to recast this force as an interaction of the moving charge with moving charges in the wire. This can be done by rewriting the above formula as an integral

$$\vec{F} = \frac{(\text{const})}{2} \; Q \cdot I \cdot \int \left(\frac{-\hat{r}\; \vec{v}\cdot d\vec{\ell} + d\vec{\ell}\; \vec{v}\cdot\hat{r}}{|\vec{r}|^2} \right)$$

where $\vec{r}$ is the vector from a point on the wire to the charge:

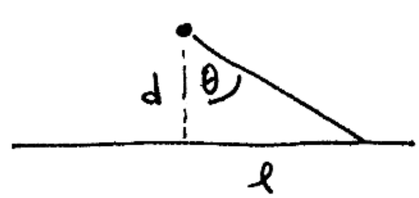


To check this, consider

$$\int d\ell^i \; \hat{r}^j \frac{1}{|\vec{r}|^2} = (\hat{n})^i \int_{-\infty}^{\infty} d\ell \; \frac{\hat{r}^j}{|r|^2}$$

The integral over the parallel component of $\hat{r}$ vanishes by symmetry. The ~~integral~~ over the perpendicular component of $\hat{r}$ is

$$(\hat{n})^i (\hat{d})^j \int_{-\infty}^{\infty} d\ell \; \frac{d}{(\ell^2+d^2)^{3/2}}$$



to evaluate the integral, write

$$\frac{l}{d} = \tan\Theta \quad \Rightarrow \quad dl = d\,\frac{d\Theta}{\cos^2\Theta}$$

$$\left(\frac{d}{(l^2+d^2)^{1/2}}\right)^3 = \cos^3\Theta$$

so the integral is

$$(\hat{n})^i\,(\hat{d})^j\;\frac{1}{d}\int_{-\pi/2}^{\pi/2} d\Theta\,\cos\Theta \;=\; \hat{n}^i\,\hat{d}^j\,\frac{2}{d}$$

Putting this into the expression on p. 4 does indeed give the formula on p. 3.

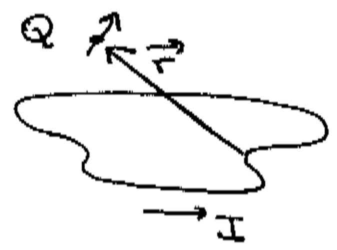
However, there is something peculiar about representing this as an interaction of individual charges. That interaction would be

$$\vec{F} = \frac{(\text{const})''}{2}\,Q.\,Q'\left(\frac{-\hat{r}\,\vec{v}\cdot\vec{v}\,' + \vec{v}\,'\,\vec{v}\cdot\hat{r}}{|\vec{r}|^2}\right)$$

But this does not reverse itself when we interchange the charges Q and Q', so it violates Newton's second law — momentum conservation! In fact, $\vec{F}$ is the correct expression for the force <u>on</u> a moving charge due to a <u>stationary</u> distribution of <u>currents</u>, but does not apply directly to the forces set up by a moving charge. For the moment, we will discuss <u>stationary currents</u> only ("magnetostatics") and postpone a full discussion of moving charges to 121.

The expression on p. 4 is actually the correct expression for the force on a moving charge due to any current distribution. It is conventionally written:

$$\vec{F} = \frac{\mu_0}{4\pi} Q I \oint \left(\frac{-\hat{r}\,\vec{v}\cdot d\vec{l} + d\vec{l}\,\vec{v}\cdot\hat{r}}{|\vec{r}|^2} \right)$$



This is the Biot Savart law (version 1). The constant μ_0 is called the "permeability of free space". It's units are N/A^2. It's value is

$$\frac{\mu_0}{4\pi} = 10^{-7}\ N/A^2 \quad (\text{exactly!})$$

The value is exact because this is the definition of the ampere. (The Coulomb is then defined as 1 ampere × 1 sec.) For a current that is distributed over space, we can replace

$$I\,d\vec{l} \longrightarrow \vec{J}\,d^3x$$

units: $C\,m/\text{sec}$ units $C/m^2\text{sec} \cdot m^3$

Then the force on a charge due to a volume current distribution is

$$\vec{F} = \frac{\mu_0}{4\pi} Q \int d^3x \; \frac{-\hat{r}\,\vec{v}\cdot\vec{J} + \vec{J}\,\vec{v}\cdot\hat{r}}{|\vec{r}|^2}$$

This expression applies to any stationary current, i.e. to any current that satisfies

$$\frac{\partial}{\partial t}\vec{J} = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{J} = 0 \;, \quad \text{so} \quad \frac{\partial \rho}{\partial t} = 0$$

In electrostatics, it was useful to factorize the expression for the force into an expression for a field created by charges and an expression for the force as the local action of this field. That can also be done here. Since $\vec{F} \perp \vec{v}$, an appropriate form for the force is

$$\vec{F} = Q\left(\vec{v} \times \vec{B}(\vec{x})\right)$$

where $\vec{B}$ is a vector field, evaluated at the location $\vec{x}$ of the charge. The $\vec{B}$ set up by a current distribution is then

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

where $\vec{r}$ is a vector from the current distribution to $\vec{x}$.

For currents in a volume

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3y \; \frac{\vec{J} \times (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}$$

This is the standard form of the Biot-Savart law.

Using
$$\vec{\nabla} \times (d\vec{l} \times \hat{r}) = d\vec{l}\; \vec{\nabla}\cdot\hat{r} - \hat{r}\,(\vec{\nabla}\cdot d\vec{l})$$

we recover our earlier formulae.

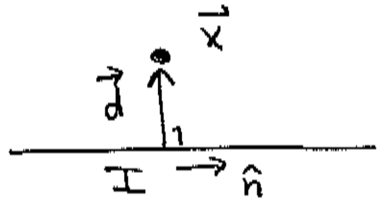
Note that the use of the standard cross product — evaluated by the right-hand rule — is purely a matter of convention. If we evaluated all cross-products by the left-hand rule, all forces would come out the same. However, $\vec{B}$ is defined in the standard way with right-hand-rule conventions. (When we come to a "fundamental" understanding of $\vec{B}$, this arbitrary choice should hopefully disappear.)

Let me now evaluate $\vec{B}$ for two simple configurations, a long wire and a current loop. In both cases, we start from

$$\vec{B} = \frac{\mu_0}{4\pi}\, I \oint \frac{d\vec{l} \times \vec{r}}{r^3}$$

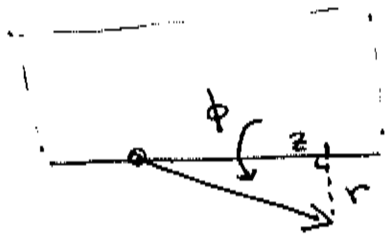
For a wire, we already saw that the component of $\vec{r}$ parallel to the wire integrates to zero, and we did the integral for the component of $\vec{r}$ perpendicular to the wire.

If $\vec{d}$ is the perpendicular vector from the wire to $\vec{x}$,



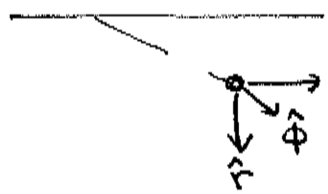
$$\vec{B} = \frac{\mu_0}{4\pi} I \cdot 2 \frac{\hat{n} \times \hat{d}}{d}$$

The vector $\hat{n} \times \hat{d}$ is a unit vector circulating around the wire in the right hand sense. In cylindrical coordinates based on the wire:



we have standard unit vectors

$$\hat{z} = (0, 0, 1)$$
$$\hat{r} = (\cos\phi, \sin\phi, 0)$$
$$\hat{\phi} = (-\sin\phi, \cos\phi, 0)$$

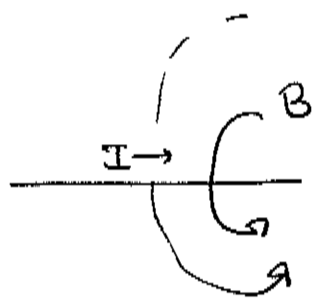


and

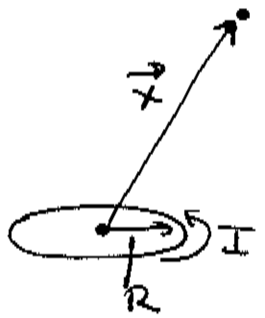
$$\hat{n} \times \hat{d} = \hat{\phi}$$

so

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{d} \hat{\phi}$$



Next, consider a current loop of radius R. Set up the loop so that its center is at $\vec{x}=0$ and so that it is oriented in the $\hat{x}, \hat{y}$ plane.



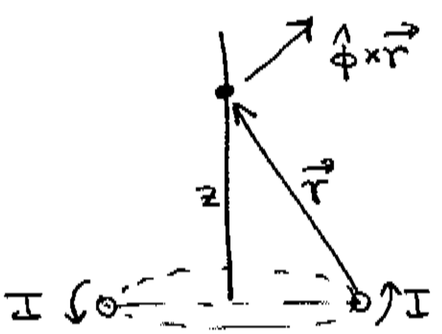
a typical point on the loop is $\vec{y} = (R\cos\phi, R\sin\phi, 0)$

$$d\vec{y} = (-R\sin\phi, R\cos\phi, 0)\,d\phi = R\,\hat{\phi}\,d\phi$$

so

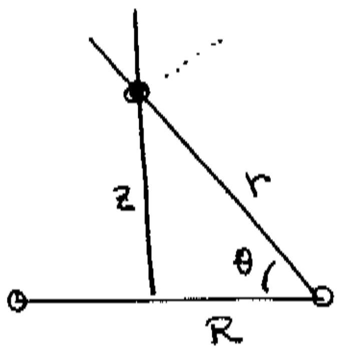
$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} I \int_0^{2\pi} d\phi\, R\, \frac{\hat{\phi}\times\vec{r}}{r^3} \qquad \text{where} \quad \vec{r} = (\vec{x}-\vec{y})$$

Let's first evaluate this <u>on the $\hat{z}$ axis</u>: $\vec{x} = (0,0,z)$



As we integrate over $d\phi$, the component of $\hat{\phi}\times\vec{r}$ that is $\perp$ to $\hat{z}$ will integrate to 0. So $\vec{B} \parallel \hat{z}$.

Let Θ be the angle of elevation of $\vec{x}$



$$r = \frac{R}{\cos\Theta}$$

and

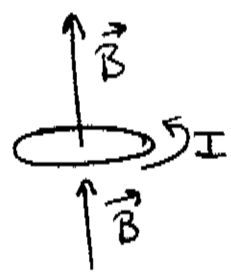
$$(\hat{\phi}\times\hat{r})^z = \cos\Theta$$

so!

$$\vec{B} = \frac{\mu_0}{4\pi} I \oint d\phi \; R \; \frac{\cos\Theta \, \hat{z}}{r^2}$$

$$= \frac{\mu_0}{4\pi} I \; \underbrace{2\pi}_{} \; R \left(\frac{\cos\theta}{R}\right)^2 \cos\Theta \, \hat{z}$$

$$\vec{B} = \frac{\mu_0 I}{2} \frac{R^2}{[z^2+R^2]^{3/2}} \hat{z}$$



We can get an idea of the rest of the $\vec{B}$ field distribution by considering the limit of a small current loop. For this, we need the expansion of $\vec{r}/r^3$ in powers of $\vec{y}$

$$\vec{r} = \overrightarrow{x-y}$$

$$r^{-3} = \left(|\vec{x}-\vec{y}|^2\right)^{-3/2}$$
$$= \left(x^2 - 2\vec{x}\cdot\vec{y} + y^2\right)^{-3/2}$$
$$= x^{-3}\left(1 + 3\,\frac{\vec{x}\cdot\vec{y}}{x^2} + \cdots\right)$$

so

$$\frac{\vec{r}}{r^3} = \frac{\vec{x}-\vec{y}}{x^3}\left(1 + 3\,\frac{\vec{x}\cdot\vec{y}}{x^2} + \cdots\right)$$

$$= \frac{\vec{x}}{x^3} + \frac{3\,\vec{x}\,(\vec{x}\cdot\vec{y}) - x^2\,\vec{y}}{x^5} + \cdots$$

since $\oint d\phi\ \hat{\phi}\times\vec{x} = 0$, the second term here gives the leading contribution:

$$\vec{B}(\vec{x}) \cong \frac{\mu_0}{4\pi}\,\frac{IR}{x^3}\int_0^{2\pi} d\phi\ \ \hat{\phi}\times\left(3\hat{x}\ \hat{x}\cdot\vec{y} - \vec{y}\right)$$

It is not hard to evaluate

$$\int_0^{2\pi} d\phi\ \ \hat{\phi}\times\vec{y} = 2\pi R\,(-\hat{z})$$

but the first term is more complicated. To approach this term, compute

$$\int_0^{2\pi} d\phi\ \ \hat{\phi}^{\,i}\,(\vec{y})^{j}$$

$$= \int_0^{2\pi} d\phi \quad (-\sin\phi, \cos\phi, 0)^i \cdot R \cdot (\cos\phi, \sin\phi, 0)^j$$

$$= \pi \cdot R \cdot \begin{cases} -1 & \text{if } i=1 \quad j=2 \\ 1 & \text{if } i=2 \quad j=1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= -\pi R \; \epsilon^{ijz} \qquad (\epsilon^{xyz} = +1 \text{ \& antisymmetric})$$

as a check

$$\int_0^{2\pi} d\phi \; (\hat{\phi} \times \vec{y})^k = \epsilon^{ijk} \int_0^{2\pi} d\phi \; \hat{\phi}^i \vec{y}^j$$

$$= \epsilon^{ijk} (-\pi R \; \epsilon^{ijz})$$

$$= -2\pi R \; \delta^{kz}$$ which agrees with the result above.

now

$$\int_0^{2\pi} d\phi \; [\hat{\phi} \times (3 \hat{x} \; \hat{x} \cdot \vec{y})]^k$$

$$= \int_0^{2\pi} d\phi \; \hat{\phi}^i \vec{y}^j \, 3\epsilon^{ilk} \hat{x}^l \hat{x}^j$$

$$= -\pi R \, \epsilon^{ijz} \; 3 \, \epsilon^{ilk} \; \hat{x}^l \hat{x}^j$$

now $\quad \epsilon^{ijz} \epsilon^{ilk} = (\delta^{jl} \delta^{kz} - \delta^{jk} \delta^{lz})$

$$= -3\pi R \, [(\hat{x} \cdot \hat{x}) \, \delta^{kz} - \hat{x}^k \hat{x}^z]$$

$$= (-3\pi R \; \hat{z} + 3\pi R \; \hat{x} \; \hat{x} \cdot \hat{z})^k$$

so

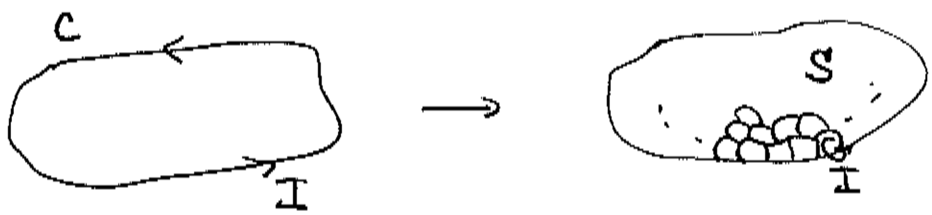
$$\vec{B} = \frac{\mu_0}{4\pi} \frac{I R}{x^3} \cdot \pi R \cdot \left(3 \hat{x}\, \hat{x} \cdot \hat{z} - \hat{z} \right)$$

Let me write this as:

$$(\vec{B})^i = \frac{\mu_0}{4\pi} \left(\frac{3 \hat{x}^i \hat{x}^j - \delta^{ij}}{x^3} \right) p^j$$

where $p^j = (I \pi R^2 \hat{z})^j$

This field has the form of an electric dipole. We might call it a magnetic dipole, with dipole moment $\vec{p}$. We can describe the $\vec{B}$ field of a large current loop by dividing it into small loops and representing each by a dipole:



then

$\hat{n}$ = normal to surface.

$$\vec{B} = \int_S d^2x \; \frac{\mu_0}{4\pi} \left(\frac{3 \hat{x}\; \hat{x} \cdot (I\hat{n}) - I\hat{n}}{x^3} \right)$$

This is a third form of the Biot-Savart law.