Boundary-Value Problems with Dielectrics

Nov. 8

In the previous lecture, we derived the electrostatic equations valid in the presence of polarizable media:

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\times\vec{E} = 0$$

where ρ_f is the "free" or explicit charge density, and $\vec{D}$ is a function of $\vec{E}$ which depends on the medium. In a linear isotropic medium

$$\vec{D} = \epsilon\vec{E}$$

I will assume linearity from here on.

If ϵ is constant $\vec{E}$ obeys

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\epsilon} \qquad \vec{\nabla}\times\vec{E} = 0$$

and these equations are identical to the usual electrostatic equations in vacuum. We can analyze these equations more generally, assuming that ϵ is a well-defined function of $\vec{x}$. Since $\vec{\nabla}\times\vec{E} = 0$, we can write

$$\vec{E} = -\vec{\nabla}\phi$$

Then

$$\vec{D} = -\epsilon(x)\vec{\nabla}\phi$$

and the equation for $\vec{D}$ becomes the generalized Poisson equation

$$-\vec{\nabla}\left(\varepsilon(x)\,\vec{\nabla}\phi\right) = \rho_f(x)$$

By a simple generalization of our argument for Poisson's equation, it is not hard to show that this equation has a unique solution for fixed Dirichlet or Neumann boundary conditions as long as $\varepsilon(x) > 0$. Let $\phi_1(x)$ $\phi_2(x)$ be two solutions to the equation satisfying the same boundary conditions. Let $\varphi = \phi_1 - \phi_2$. Then

$$-\vec{\nabla}(\varepsilon\vec{\nabla}\varphi) = 0$$

and $\varphi = 0$ or $\hat{n}\cdot\vec{\nabla}\varphi = 0$ on the boundary. Now consider

$$\mathcal{J} = \int d^3x \;\; \varepsilon(x)\,(\vec{\nabla}\varphi)^2$$

If $\varepsilon(x) > 0$, $\mathcal{J} > 0$ and $\mathcal{J} = 0$ only if $\varphi = 0$ identically.
But

$$\mathcal{J} = \int d^2x\; \hat{n}\cdot\varepsilon(x)\,\varphi\,\vec{\nabla}\varphi \;-\; \int d^3x\;\varphi\,\vec{\nabla}(\varepsilon\vec{\nabla}\varphi)$$

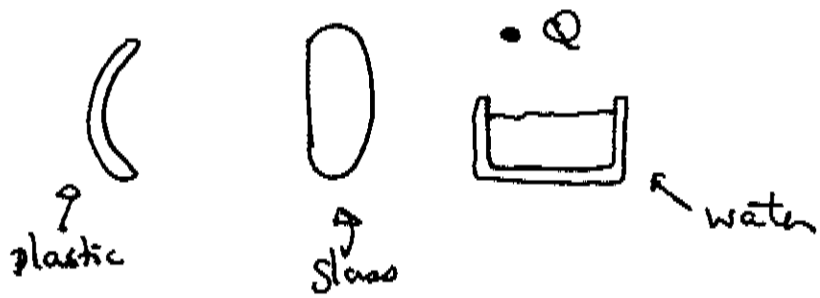
$$= \qquad 0 \qquad + \qquad 0$$

by the conditions on φ listed above. So $\varphi = 0$.
An ~~interesting~~ aspect of this theorem is that it depends on an assumed behavior of the material. Note that if $\chi_e > 0$,

$\varepsilon > \varepsilon_0 > 0$ and so the required property is satisfied.

If a material does not have this "sensible" property, the electrostatic equations can allow weird instabilities.

It is not easy to solve the electrostatic equations for a general function $\varepsilon(x)$. However, it is often the case that $\varepsilon(x)$ is constant in a given region, though it varies from region to region. This is the case if we consider a collection of different materials whose shapes are fixed

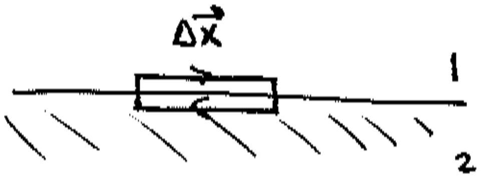


Inside each medium, Φ solves Laplace's equation. But, we need to think about how to glue these solutions together at the boundaries.

To work this out, we can use the fact that the electrostatic equations are actually true just at the boundaries, even though $\varepsilon(x)$ changes discontinuously there. So let's make use of this. First consider

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \oint_C d\vec{l} \cdot \vec{E} = 0$$

apply this to the contour:



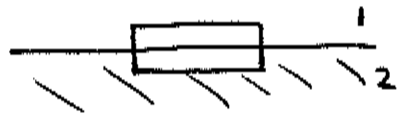
$$\oint_C d\vec{l}\cdot\vec{E} = \Delta\vec{x}\cdot(\vec{E}_1 - \vec{E}_2)$$

so we find that, at the bondy, $\vec{E}_{\parallel}\big|_1 = \vec{E}_{\parallel}\big|_2$

Next consider

$$\vec{\nabla}\cdot\vec{D} = \rho_f \quad\Rightarrow\quad \int_S d^2x\, \hat{n}\cdot\vec{D} = Q_f$$

apply this to the surface



$$\int d^2x\, \hat{n}\cdot\vec{D} = (\text{Area})\cdot \hat{n}\cdot(\vec{D}_1 - \vec{D}_2) = (\text{Area})\cdot \underbrace{\rho_{f,s}}_{\text{surface density of free charge.}}$$

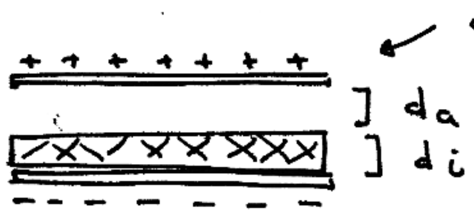
so

$$D_\perp\big|_1 = D_\perp\big|_2 + \rho_{f,s}$$

which implies a discontinuity in $E_\perp$ if $\varepsilon_1 \neq \varepsilon_2$. At a surface with no explicit or free charge

$$\vec{E}_{\parallel}\big|_1 = \vec{E}_{\parallel}\big|_2 \qquad D_\perp\big|_1 = D_\perp\big|_2$$

I will now discuss a series of examples that illustrate the use of this formalism: Let's begin with a simple example: A large parallel-plate capacitor



partly filled with an insulator with permittivity ϵ.
Just below the top surface, in air or vacuum,

$$E = \frac{\rho}{\epsilon_0} \qquad D = \rho$$

$\vec{E}, \vec{D}$ are normal to the surfaces. At the interface between air and insulator, $D_\perp$ is continuous, so in the insulator

$$D = \rho$$

(this is also what is required by the surface charge distribution at the bottom) and

$$E = \frac{\rho}{\epsilon} \qquad \text{(smaller than in air!)}$$

The potential difference across the air gap is

$$\Delta\varphi = \frac{\rho}{\epsilon_0} d_a$$

The potential difference across the insulator is

$$\Delta\varphi = \frac{\rho}{\epsilon} d_i$$

The total potential difference is $\frac{\rho}{\epsilon_0} d_a + \frac{\rho}{\epsilon} d_i$. The charge on the capacitor is

$$A\rho$$

where A is the area, so

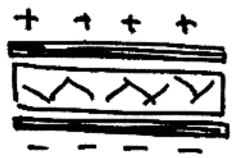
$$C = \frac{Q}{V} = \frac{A}{\left(\frac{d_a}{\epsilon_0} + \frac{d_i}{\epsilon}\right)}$$

Now contrast a capacitor with air inside and a capacitor with dielectric inside:

	air	dielectric
capacitance	$C = \frac{\epsilon_0 A}{d}$	$C = \frac{\epsilon A}{d}$
energy stored at fixed V $E = \frac{1}{2}CV^2$	$E = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2$	$E = \frac{1}{2}\frac{\epsilon A}{d}V^2$
energy stored at fixed Q $E = \frac{1}{2}Q^2/C$	$E = \frac{1}{2}\frac{d}{\epsilon_0 A}Q^2$	$E = \frac{1}{2}\frac{d}{\epsilon A}Q^2$

The capacitor with dielectric has a larger capacitance, that is, it stores more charge for the same potential difference.

At fixed Q, the capacitor with dielectric stores less energy. So, if we have a capacitor unhooked from a battery



it will suck in the dielectric, and we must do work to get the dielectric out. What if the capacitor is connected to a battery enforcing fixed voltage V. Then the energy is greater with the dielectric in:

$$E_{diel.} - E_{air} = \frac{1}{2} \frac{(\epsilon - \epsilon_0) A}{d} V^2$$

However, the capacitor with dielectric stores more charge. To move this charge from one plate to the other, the battery does work

$$W = \Delta Q \, V = \left(\frac{\epsilon A}{d} V - \frac{\epsilon_0 A}{d} V \right) \cdot V$$

$$= \frac{(\epsilon - \epsilon_0) A}{d} V^2$$

This is twice as much work as the final difference in energy.
So again the work done on the dielectric is negative

$$W_{diel} = -\frac{1}{2}\frac{(\epsilon-\epsilon_0)A}{d}V^2$$

again, the dielectric is sucked in.

Let's now compute the energy stored in the electric field in the presence of more general assemblies of dielectric material:

I want to compute the work needed to assemble this distribution from pieces placed at ∞. We can do the assembly in any order. Here is a convenient one!

First, bring in the pieces of dielectric material. Since there are no $\vec{E}$ fields around, this takes no work.

Now bring in the explicit charges, a little at a time.
As each charge is brought in, we must do work

$$\Delta W = \Delta Q \;\phi(\vec{r})$$

where ΔQ is the amount of charge brought in from infinity and $\phi(\vec{r})$ is the electrostatic potential at the place where we put it. $\phi(\vec{r})$ depends on the charge already in place. If we build up the final distribution of charge $\rho(\vec{x})$ uniformly.

$$dQ = d\alpha \;\rho(\vec{x}) \qquad 0 \le \alpha < 1$$

then

$$dW = \int d^3x \;\; d\alpha \;\rho(x) \cdot \phi_\alpha(\vec{x})$$

where $\quad \phi_\alpha(\vec{x}) = \alpha\, \phi(\vec{x})$

if $\phi(\vec{x})$ is the <u>final</u> potential. Then the total work is

$$W = \int_0^1 d\alpha \int d^3x \;\; \rho(x) \;\; \alpha\, \phi(\vec{x})$$

$$= \frac{1}{2} \int d^3x \;\; \rho(\vec{x})\, \phi(\vec{x})$$

This looks just like the formula we derived in our earlier discussion of electromagnetic energy. But, here, $\rho(\vec{x})$ is the explicit charge density – the free charge. So, we can rewrite this as.

$$W = \frac{1}{2}\int d^3x \; \vec{\nabla}\cdot\vec{D} \; \phi(x)$$

$$= \frac{1}{2}\int d^3x \; \vec{D}\cdot(-\vec{\nabla}\phi) \quad \left(\begin{array}{l}\text{plus a surface term at} \\ \infty \text{ which is negligible}\end{array}\right)$$

$$= \frac{1}{2}\int d^3x \; \vec{D}\cdot\vec{E}$$

So in the presence of dielectrics, the electrostatic energy stored in fields is

$$E = \frac{1}{2}\int d^3x \; \vec{D}\cdot\vec{E}$$

since $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$,

$$E = \underbrace{\frac{1}{2}\int d^3x \; \epsilon_0 E^2}_{\substack{\text{this is the energy in} \\ \text{macroscopic } \vec{E} \\ \text{fields}}} + \underbrace{\frac{1}{2}\int d^3x \; \vec{P}\cdot\vec{E}}_{\substack{\text{this is the energy stored} \\ \text{in microscopic "springs"} \\ \text{which oppose the creation} \\ \text{of a polarization.}}}$$

The second term is the work required to polarize the dielectrics.

Let's check this formula against our results for a parallel-plate capacitor filled with dielectric.

For the parameters on p. 6, V fixed

$$E = \frac{V}{d} \qquad D = \epsilon \frac{V}{d}$$

$$\text{Energy} = \frac{1}{2}\int d^3x \; \vec{D}\cdot\vec{E} = \frac{1}{2}(Ad)\left(\epsilon\frac{V}{d}\right)\left(\frac{V}{d}\right)$$

$$= \frac{1}{2}\,\frac{\epsilon A}{d}\,V^2$$

in agreement w. p. 6.

From these energetic arguments, we can obtain the intuition that electrostatic fields can lower their energy by being sucked into a dielectric. I'd like now to analyze two problems which illustrate this and, at the same time illustrate solutions of electrostatic problems with dielectrics.

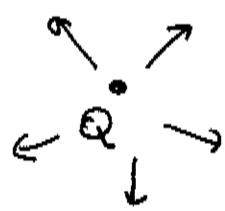
It is easy to solve the problem of a point charge in a dielectric. The $\vec{E}$ field is radial and symmetric

from

$$\vec{\nabla}\cdot\vec{D} = \rho_f = Q\,\delta^{(3)}(\vec{r})$$

we find

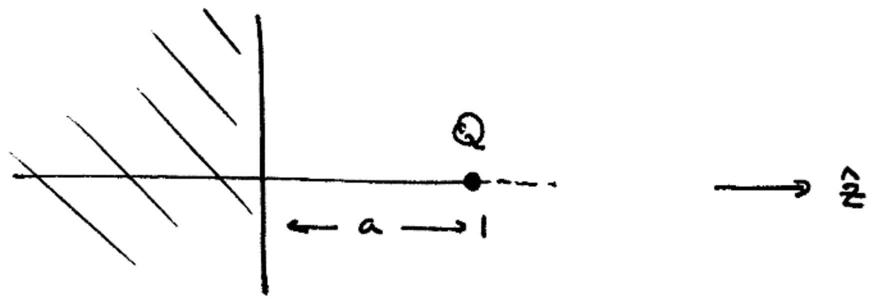
$$\vec{D} = \frac{Q}{4\pi}\frac{\hat{r}}{r^2}$$



and

$$\vec{E} = \frac{Q}{4\pi\epsilon}\frac{\hat{r}}{r^2}$$

now, what if we have a point charge at a point near an interface of two dielectric media.



For simplicity, take the medium on the right to be vacuum. The medium on the left has permittivity $\epsilon = \epsilon_0 (1 + \chi_e)$.

In a similar problem with a conductor, we found the solution by the method of images. This problem can also be solved by the method of images. I'll first give the solution and then discuss its physics.

We have to solve the electrostatic equation in each of two regions, left and right. In each region, the electrostatic potential φ satisfies the Laplace or Poisson equation. Let's construct appropriate solutions, and then knit them together using the boundary conditions on E and D.

In the right region, write

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left\{ \frac{Q\,(\vec{r}-\vec{z}_+)}{(\vec{r}-\vec{z}_+)^3} - \frac{Q'\,(\vec{r}-\vec{z}_-)}{(\vec{r}-\vec{z}_-)^3} \right\} \qquad \underline{\underline{z>0}}$$

where $\vec{z}_+ = (0,0,a)$; $z_- = (0,0,-a)$ is the image point. The strength of the image charge Q' will be determined later.

In the left region, write

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q'' \, (\vec{r}-\vec{z}_+)}{(\vec{r}-\vec{z}_+)^3} \qquad \underline{z<0}$$

The corresponding potential satisfies $-\nabla^2\varphi = 0$ in the region $z<0$.
Now evaluate the E fields at the boundary. A typical boundary point is $\vec{r} = (x,0,0)$ at this point

right: $\underline{z=0^+}$

$$E_{\parallel} = \frac{1}{4\pi\epsilon_0}\left[\frac{Q\,x}{(x^2+a^2)^{3/2}} - \frac{Q'\,x}{(x^2+a^2)^{3/2}}\right]$$

$$E_{\perp} = \frac{-1}{4\pi\epsilon_0}\left[\frac{Q\,a}{(x^2+a^2)^{3/2}} + \frac{Q'\,a}{(x^2+a^2)^{3/2}}\right]$$

left: $\underline{z=0^-}$

$$E_{\parallel} = \frac{1}{4\pi\epsilon_0}\frac{Q''x}{(x^2+a^2)^{3/2}}$$

$$E_{\perp} = \frac{-1}{4\pi\epsilon_0}\frac{Q''a}{(x^2+a^2)^{3/2}}$$

so the functional forms are correct for the boundary conditions to match if we set

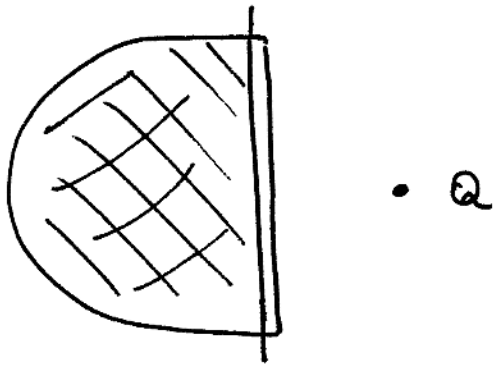
$$E_{\parallel}|_{left} = E_{\parallel}|_{right} \Rightarrow \qquad Q - Q' = Q''$$

$$D_{\perp}|_{left} = D_{\perp}|_{right} \Rightarrow \qquad \epsilon_0(Q+Q') = \epsilon Q''$$

We have two equations for two unknowns Q', Q''. The solution is: $Q' = \dfrac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}Q \qquad Q'' = \dfrac{2\epsilon_0}{\epsilon+\epsilon_0}Q$

Notice that, for $\epsilon = \epsilon_0$, $Q' = 0$ and $Q'' = Q$.
This is a check on our analysis.

Since $\nabla\cdot E|_{\text{left}} = 0$ for $z<0$ and D, E, P are all proportional, $\vec{\nabla}\cdot\vec{P} = 0$ inside the medium, so all of the bound charge lies on the surface at $z=0$. Using the surface

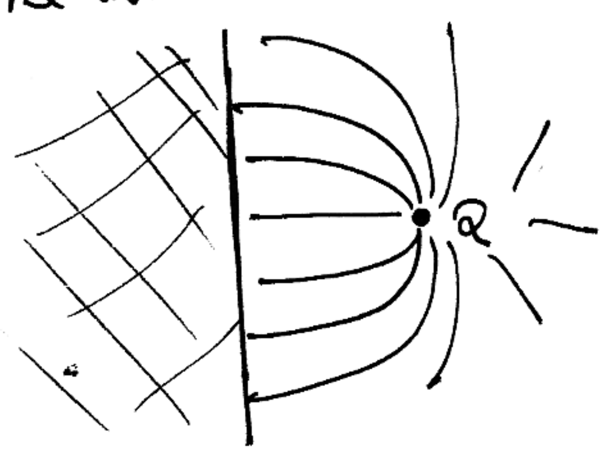


we can evaluate

$$\int d^2x\, \hat{n}\cdot\vec{E} = \frac{\text{total charge enclosed}}{\epsilon_0} = \frac{Q'}{\epsilon_0}$$

so
$$\int d^2x \underbrace{\rho_{b,s}}_{\text{surface bound charge.}} = Q'$$

as Q' increases, more and more of the total flux of Q is sucked into the dielectric:



we can evaluate this surface charge from the discontinuity of $\vec{E}$ at the surface:

$$\rho_{b,s} = \epsilon_0\left(\hat{n}\cdot\vec{E}\big|_{right} - \hat{n}\cdot\vec{E}\big|_{left}\right)$$

$$= \epsilon_0\left(-\frac{1}{4\pi\epsilon_0}\right)\frac{a}{(x^2+a^2)^{3/2}}\cdot\left(Q+Q'-Q''\right)$$

$$= -\frac{1}{4\pi}\frac{a}{(x^2+a^2)^{3/2}}\left(Q+\left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)Q-\frac{2\epsilon_0}{\epsilon+\epsilon_0}Q\right)$$

$$= -\frac{1}{4\pi}\frac{2(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}\frac{a}{(x^2+a^2)^{3/2}}Q$$

$$\rho_{b,s} = -\frac{1}{2\pi}\left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)\frac{a}{(x^2+a^2)^{3/2}}Q = -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{a}{(x^2+a^2)^{3/2}}Q$$

There is actually another way to obtain this result: We can compute the $\vec{E}$ on the surface self-consistently with the $\vec{P}$ or $\rho_{b,s}=\hat{n}\cdot\vec{P}$ that it produces. Start with

$$\rho_{b,s} = \hat{n}\cdot\vec{P} = \hat{n}\cdot(\epsilon_0\chi_e\vec{E})$$

now, $\vec{E}$ comes from two sources: First, there is the $\vec{E}$ due to the point charge; second, there is the $\vec{E}$ set up by the surface charge. Since the surface is flat, surface charges far away from a given point $\vec{r}=(x,0,0)$ cannot contribute to $\vec{E}_\perp$

so just inside the surface

$$\hat{n}\cdot\vec{E} = \underbrace{\frac{-1}{4\pi\epsilon_0}\frac{Q\,a}{(x^2+a^2)^{3/2}}}_{\text{point charge}} + \underbrace{\frac{(-\rho_{b,s})}{2\epsilon_0}}_{\text{surface layer.}}$$

putting the pieces together!

$$\rho_{b,s} = \epsilon_0\chi_e\left(-\frac{1}{4\pi\epsilon_0}\frac{Q\,a}{(x^2+a^2)^{3/2}} - \frac{\rho_{bs}}{2\epsilon_0}\right)$$

$$(2+\chi_e)\,\rho_{b,s} = -\frac{\chi_e}{2\pi}\frac{Q\,a}{(x^2+a^2)^{3/2}}$$

$$\rho_{b,s} = -\frac{1}{2\pi}\frac{\chi_e}{2+\chi_e}\frac{a}{(x^2+a^2)^{3/2}}\,Q$$

just as above.

Finally, what is the force on the point charge. This force is due to the part of the $\vec{E}$ field acting on the charge that is not due to the charge itself. This is.

$$\vec{F} = Q\cdot\left(\frac{-1}{4\pi\epsilon_0}\frac{Q'\,(\vec{r}-\vec{z}_-)}{(r-z_-)^3}\right)\Bigg|_{\vec{r}=z_+}$$

$$\vec{F} = -\frac{QQ'}{4\pi\epsilon_0}\frac{1}{4a^2}\,\hat{z}$$

$$= -\frac{1}{16\pi\epsilon_0}\frac{Q^2}{a^2}\left(\frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0}\right)\hat{z}$$

Here is another model problem that illustrates a different method for solving the electrostatic equations: Consider a dielectric sphere of permittivity ϵ and radius R in a background $\vec{E}$ field $E_0\hat{z}$. What is the polarization of the sphere, and what fields does it produce?

To solve this problem, we can simply make use of the fact that φ satisfies Laplace's equation in each region. For $r < R$, the most general cylindrically symmetric solution of Laplace's equation has the form

$$\varphi_{in} = \sum_{l=0}^{\infty} A_l \, r^l \, P_l(\cos\theta)$$

For $r > R$, the most general cylindrically symmetric solution of Laplace's equation has the form

$$\varphi_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) - E_0 r \cos\theta$$

I have used the constraint that, as $r \to \infty$, the $\vec{E}$ field should tend to $E_0 \hat{z}$ and so $\varphi \to -E_0 z$.

Now we can simply match these forms at the boundary. We need

$$\vec{E} = -\vec{\nabla}\varphi = -\hat{r}\frac{\partial}{\partial r}\varphi - \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\varphi$$

so $$E_{\parallel} = -\frac{1}{r}\frac{\partial}{\partial\theta}\varphi \qquad E_{\perp} = -\frac{\partial}{\partial r}\varphi$$

then

<u>in:</u>

$$E_{\parallel} = -\sum_{\ell=0}^{\infty} A_\ell r^{\ell-1} \frac{\partial}{\partial\theta} P_\ell(\cos\theta)\Big|_{r=R}$$

$$E_{\perp} = -\sum_{\ell=0}^{\infty} A_\ell \cdot \ell\, r^{\ell-1} P_\ell(\cos\theta)\Big|_{r=R}$$

<u>out:</u>

$$E_{\parallel} = -\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+2}} \frac{\partial}{\partial\theta} P_\ell(\cos\theta)\Big|_{r=R} + E_0 \frac{\partial}{\partial\theta} P_1(\cos\theta)$$

$$E_{\perp} = -\sum_{\ell=0}^{\infty} -\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\Big|_{r=R} + E_0 P_1(\cos\theta)$$

so we can match the boundary conditions by setting

$$A_\ell = B_\ell = 0 \qquad \text{for } \ell \neq 1$$

and, for $\ell = 1$:

$$E_{\parallel}\big|_{in} = E_{\parallel}\big|_{out} \Rightarrow \qquad A_1 = \frac{B_1}{R^3} - E_0$$

$$D_{\perp}\big|_{in} = D_{\perp}\big|_{out} \Rightarrow \qquad \epsilon A_1 = \epsilon_0\left(-\frac{2B_1}{R^3} - E_0\right)$$

We have two equations for two unknowns. The solution is

$$A_1 = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 \qquad B_1 = \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} E_0 R^3$$

The $\vec{E}$ field inside the sphere is

$$\varphi_{in} = -\frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 \, r\cos\theta \quad \Rightarrow \quad \vec{E}_{in} = \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 \hat{z}$$

This goes back to $E_0\hat{z}$ when $\epsilon \to \epsilon_0$ but otherwise is less than E_0 (But $D_{in} > \epsilon_0 E_0$). Also $\vec{E}$, $\vec{D}$, $\vec{P}$ are uniform inside the sphere. The total dipole moment of the sphere can be found from the exterior solution

$$\varphi_{out} = -E_0 z + \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right) E_0 R^3 \frac{1}{r^2}\cos\theta$$

$$= -E_0 z + 4\pi R^3 \epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right) \frac{1}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}$$

$$= -E_0 z + \frac{1}{4\pi\epsilon_0} \vec{P} \cdot \frac{\hat{r}}{r^2}$$

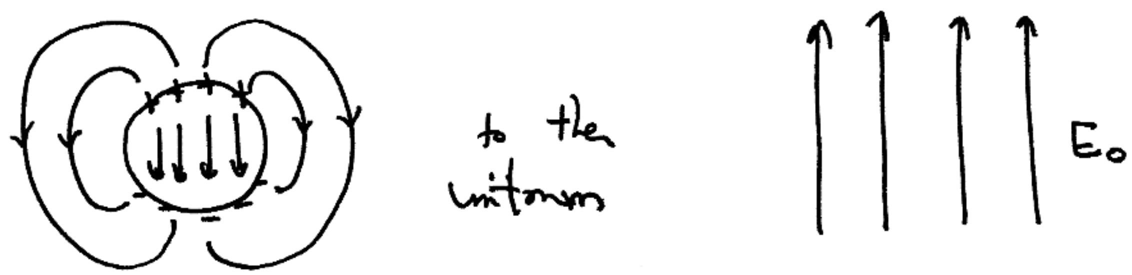
with $\quad \vec{p} = 4\pi R^3 \epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right) = \frac{4\pi R^3}{3}\left(\frac{3\chi_e}{3+\chi_e}\right)\epsilon_0 E_0$

Since the sphere is uniformly polarized:

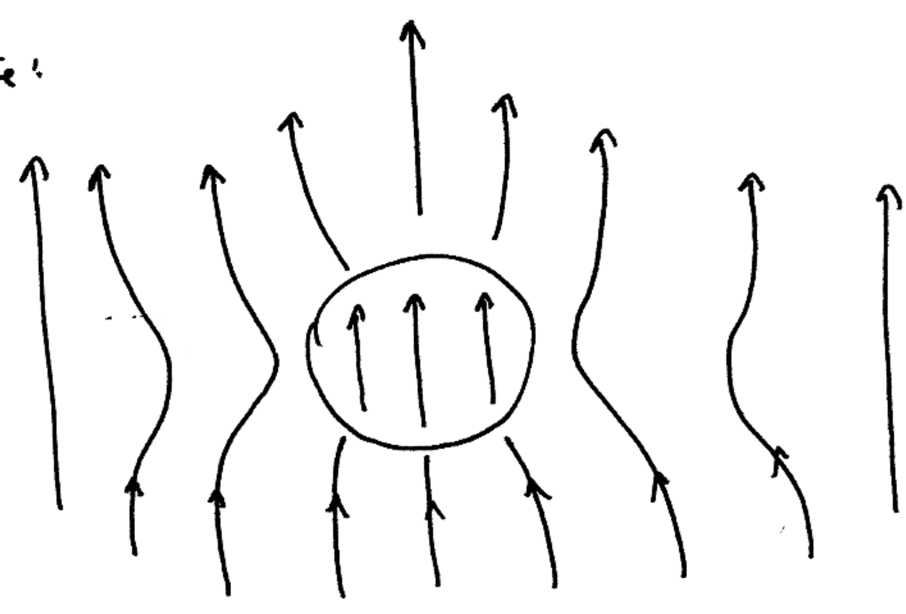
$$\vec{P} = \epsilon_0 \chi_e \left(\frac{3}{3+\chi_e}\right) E_0$$

which is just $\quad = \epsilon_0 \chi_e E_{in} \quad !$

The final situation adds a dipole field



to give:



with small $\vec{E}$, large $\vec{D}$, inside.