Polarizable Media

Nov. 1

So far in this course, the only materials we have dealt with have been conductors. What about insulators? In an insulator, charge does not flow freely, and a charge placed inside the medium is not neutralized. However, this does not mean that an insulating material does not interact with an electric field. An electric field can polarize a neutral atom or molecule, pushing the positive charges one way and the negative charges the other way

and creating a small dipole moment. If each atom in a material is polarized in this way, the whole material is characterized by a density of polarization

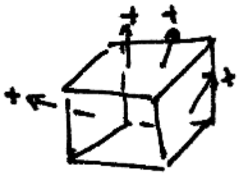
$$\vec{P} = \text{dipoles}/\text{volume} = \text{C-m}/\text{m}^3$$

A material that acquires a polarization in an electric field is called a *dielectric*.

To warm up for our study of dielectrics, let's study the properties of $\vec{P}$. To get a clean picture, model $\vec{P}$ as a density n (dipoles/m) of dipoles with charge q and charge separation $\vec{d}$

$$\vec{P} = q\vec{d}\,n$$

If we draw a small cube of side a in this medium,



+ charges and − charges may be sticking out of the cube. The net number of charges sticking through a surface is

$$\text{charge} = a^2\,\hat{n}\cdot\vec{d}\,q\,n \qquad (\text{units} = C\,!)$$

$$= \int d^2s\;\hat{n}\cdot\vec{P}$$

summing over all six sides

$$(\text{charge sticking out}) = -(\text{charge inside cube})$$

$$= \int d^2s\;\hat{n}\cdot\vec{P} = \int d^3x\;\vec{\nabla}\cdot\vec{P}$$

so, in this model, a medium with a non-uniform polarization has a local charge density

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

This charge is not free to move in the material but, rather, is bound to atoms; net charge results from the non-uniform shifting of charge within atoms. ρ_b is called the "bound charge".

Here is another derivation of the expression for bound charge. The potential due to a dipole at $\vec{x}_0$ is

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\vec{p}\cdot\frac{\widehat{r-x_0}}{|\vec{r}-\vec{x}_0|^2}$$

$$= -\frac{1}{4\pi\epsilon_0}\vec{p}\cdot\frac{\partial}{\partial\vec{r}}\left(\frac{1}{|\vec{r}-\vec{x}_0|}\right) \ \underline{\text{or}}\ -\frac{1}{4\pi\epsilon_0}\vec{p}\cdot\vec{\nabla}_r\frac{1}{|\vec{r}-\vec{x}_0|}$$

$$= +\frac{1}{4\pi\epsilon_0}\vec{p}\cdot\frac{\partial}{\partial\vec{x}_0}\left(\frac{1}{|\vec{r}-\vec{x}_0|}\right) \ \underline{\text{or}}\ \frac{+1}{4\pi\epsilon_0}\vec{p}\cdot\vec{\nabla}_{x_0}\frac{1}{|\vec{r}-\vec{x}_0|}$$

Then the potential due to the dipoles in a polarizable medium with polarization $\vec{P}(x)$ is

$$\varphi(\vec{r}) = \int d^3x\ \vec{P}(x)\cdot\frac{1}{4\pi\epsilon_0}\left(\frac{\partial}{\partial\vec{x}}\right)\left(\frac{1}{|\vec{r}-\vec{x}|}\right)$$

$$= \int d^3x\left\{\vec{\nabla}_x\left[\frac{1}{4\pi\epsilon_0}\vec{P}\,\frac{1}{|\vec{r}-\vec{x}|}\right] - \frac{1}{4\pi\epsilon_0}\vec{\nabla}\cdot\vec{P}(x)\,\frac{1}{|\vec{r}-\vec{x}|}\right\}$$

$$\varphi(\vec{r}) = \int d^2x \, \hat{n} \cdot \frac{1}{4\pi\epsilon_0} \vec{P} \, \frac{1}{|\vec{r}-\vec{x}|}$$

$$+ \int d^3x \, \frac{1}{4\pi\epsilon_0} (-\vec{\nabla}\cdot\vec{P}) \frac{1}{|\vec{r}-\vec{x}|}$$

the first integral is taken over a surface at ∞. If the polarizable medium is of finite size, $\vec{P}=0$ for $|\vec{r}|>R$ for some R. Then

$$\varphi(\vec{r}) = \int d^3x \, \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{x}|} (-\vec{\nabla}\cdot\vec{P})$$

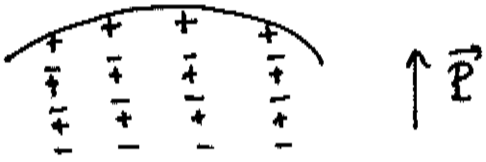
which is the potential due to a bound charge distribution

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

Note that the bound charge can be singular on surfaces. If we have a block of polarizable medium with uniform polarization $\vec{P}$ of finite size

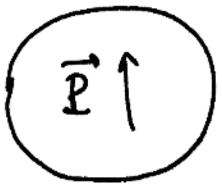
$\vec{\nabla}\cdot\vec{P} = 0$ in the interior but the charges sticking out at the surface contribute

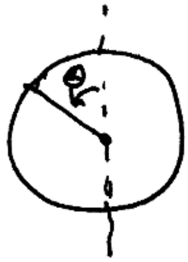
$$\text{charge} = \int d^2x \, \hat{n}\cdot\vec{P}$$



concentrated in a surface layer.

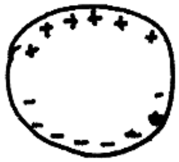
Here is an ~~interesting~~ example. Consider a uniformly polarized sphere:





The bound charge is a surface layer

$$\rho = \hat{n}\cdot\vec{P} = P\cos\theta$$



This is actually a charge distribution that we have seen before. Look back at the problem of a conducting sphere in a uniform background electric field $\vec{E} = E_0\hat{z}$

The electric field near the sphere was:

outside

$$E^x = E_0\frac{3xz R^3}{r^5} \qquad E^y = E_0\frac{3yz R^3}{r^5}$$

$$E^z = E_0 - E_0\frac{(r^2-3z^2)R^3}{r^5}$$

inside

$$\vec{E} = 0$$

This must be a superposition of the uniform field and the field due to the surface chge. We computed this surface chge and found

$$\rho = 3E_0 \epsilon_0 \cos\Theta$$

so the distribution is $\propto \cos\Theta$, just as in our new problem. To find the field due to this chge, subtract the background field from the expression for $\vec{E}$ on the previous page:

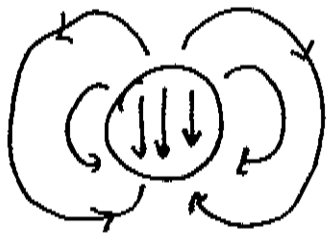
outside
$$\left[\begin{array}{ll} E^x = 3E_0 \frac{xz R^3}{r^5} & E^y = 3E_0 \frac{yz R^3}{r^5} \\ & E^z = -E_0 \frac{r^2 - 3z^2}{r^5} R^3 \end{array}\right.$$

inside
$$\left[\; E^x = E^y = 0 \qquad E^z = -E_0\right.$$

Rescaling to the chge density of the polarized sphere, we find ~~for that~~ case:

<u>outside</u>:
$$\left[\begin{array}{ll} E^x = P/\epsilon_0 \frac{xz R^3}{r^5} & E^y = \frac{P}{\epsilon_0} \frac{yz R^3}{r^5} \\ E^z = -\frac{P}{\epsilon_0} \frac{\frac{1}{3}r^2 - z^2}{r^5} R^3 & \end{array}\right.$$

<u>inside</u>:
$$\left[\; E^x = E^y = 0 \qquad E^z = -\frac{P}{3\epsilon_0}\right.$$



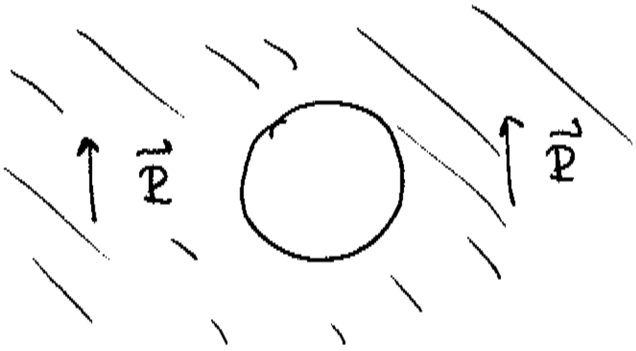
Just as a check, we can obtain the ~~potential~~ outside by rescaling our previous result

$$\varphi = \frac{P}{3\epsilon_0}\,\frac{R^3 z}{r^3}$$

$$= \frac{1}{4\pi\epsilon_0}\,\vec{p}\cdot\frac{\hat{r}}{r^3}$$

where $\vec{p} = \frac{4\pi}{3}R^3 P\,\hat{z} \overset{\checkmark}{=} \int_{\text{sphere}} d^3x\,\vec{P}$

In a similar way, you can see ~~that~~ if we have a uniformly polarized medium and we cut a spherical hole in it



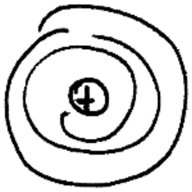
there is a surface charge of $-P\cos\theta$ on the boundary of the hole, and this produces an $\vec{E}$ field

$$\vec{E} = +\frac{\vec{P}}{3\epsilon_0}$$

inside.

Now let's apply these considerations to think about

the polarization of a real material made out of atoms.
If we put a neutral atom into an $\vec{E}$ field, the field will push the electrons one way and the nucleus the other way. However, this effect is opposed by the Coulomb attraction of the electrons and the nucleus. We can estimate the displacement of the electrons by a simple model: Think just about the last electron, the one furthest out from the nucleus. This electron is attracted by a net charge $(+e)$. Assume that this electron is distributed in a sphere of size a $(a \approx 1\,\mathring{A} = 10^{-10}\,m)$



$$\rho = (-e)\,\frac{1}{\left(\frac{4\pi}{3}a^3\right)} =$$ charge density in the sphere

If we displace this sphere by a distance d from the center, the force on the nucleus is

$$|\vec{F}| = e \cdot \left(e\,\frac{d^3}{a^3}\right)\left(\frac{1}{4\pi\epsilon_0 d^2}\right) = \frac{e^2 d}{4\pi\epsilon_0 a^3}$$

Balancing this against an external E field E_0

$$\frac{e^2 d}{4\pi\epsilon_0 a^3} = e E_0$$

we find $$d = \frac{4\pi\epsilon_0 a^3}{e} E_0$$

That is, the atom acquires a dipole moment $\vec{p} = e\vec{d}$ given by

$$\vec{p} = 4\pi\epsilon_0 a^3 \vec{E}$$

Typical electric fields in an atom are of size

$$E_{at} \sim \frac{e}{4\pi\epsilon_0 a^2} \qquad a \sim 1\,\text{Å}$$

$$\sim \left(10^{10} \frac{N m^2}{C^2}\right) \frac{10^{-19} C}{(10^{-10} m)^2} \sim 10^{11}\, V$$

so the puny electric fields we apply in the lab distort the atom only a tiny amount. It is thus reasonable that the dipole moment induced by an external $\vec{E}$ is, in any event, linear in the perturbing $\vec{E}$:

$$\vec{p} = \alpha \vec{E}$$

We have ~~estimated~~ $\alpha \sim 4\pi\epsilon_0 \cdot (1\text{Å})^3$; in reality

$\frac{\alpha}{4\pi\epsilon_0 \cdot (1\text{Å})^3} =$	0.67	0.205	24.3	…	0.40	24.1	…
	H	He	Li		Ne	Na	

So this crude estimate is not too bad for atoms with all electrons strongly bound, though it fails miserably for atoms ~~that~~ ionize readily.

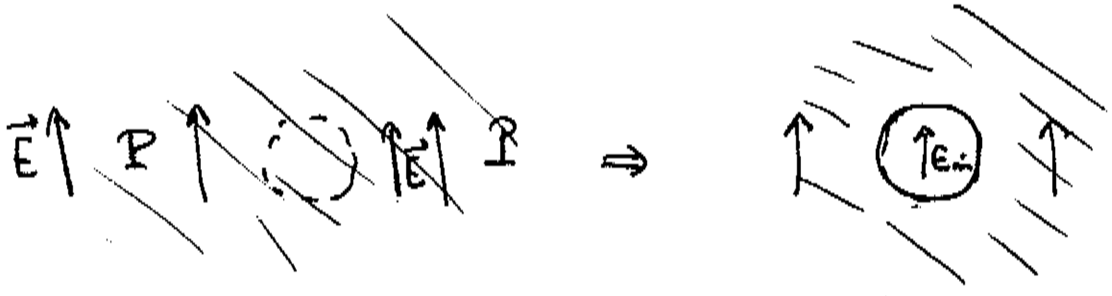
Now build up a crystal of these atoms. Inside the material, there will be some electric field, and the atoms will polarize in response to it. If $\vec{E}_{in}$ is the electric field felt by a given atom, that atom will polarize to give a dipole moment

$$\vec{p} = \alpha \vec{E}_{in}$$

and the material as a whole will acquire a polarization

$$\vec{P} = n\alpha \vec{E}_{in}$$

where n is the density of atoms. To understand this better, a crude model is again helpful. Picture the material as a uniform polarizable medium with polarization $\vec{P}$ and electric field $\vec{E}$. Picture an atom as a polarizable sphere of size a in this medium.



Part of the E field $\vec{E}$ is produced by the polarization. We shouldn't count this part when we assess the field acting on the atom. To take account of this, remove the atom from the medium. The removal of this sphere gives an extra field in the empty cavity of $+\frac{1}{3\epsilon_0}\vec{P}$. Thus

$$\vec{E}_{in} = \vec{E} + \frac{1}{3\epsilon_0}\vec{P}$$

$$\vec{p} = \alpha\left(\vec{E} + \frac{1}{3\epsilon_0}\vec{P}\right)$$

$$\vec{P} = n\alpha\left(\vec{E} + \frac{1}{3\epsilon_0}\vec{P}\right)$$

we can solve this equation for $\vec{P}$:

$$\vec{P} = \left(\frac{n\alpha}{1 - \frac{n\alpha}{3\epsilon_0}}\right)\vec{E}$$

This is the Clausius-Mossotti equation (version 1). This equation is often taken more seriously because it can be derived in another way: cut a large hole in a polarizable medium, with the atom in question at the center. Again, the atom feels a field

$$\vec{E}_{in} = \vec{E} + \frac{1}{3\epsilon_0}\vec{P}$$

from the overall E field and the bound charges around the hole. Then put back the other atoms in the hole, one spherical shell at a time. Each spherical shell of dipoles contributes *zero* field at the center. This is also true of the atoms we put back the at the sites of a cubic lattice (however, it is not true for a more general ordered configuration).

Again, since typical applied fields are much smaller than typical atomic fields, we expect that $\vec{P}$ will be proportional to $\vec{E}$. So, in general for such small $\vec{E}$,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

χ_e is a constant called the *electric susceptibility*

There is a neat way to account for $\vec{P}$ in the electrostatic equations. Recall that

$$\vec{\nabla}\times\vec{E} = 0 \qquad\qquad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

write $\rho = \rho_f + \rho_b$. ρ_f, the "free charge" are explicit charges placed in the medium. ρ_b is the bound charge

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

Then

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\epsilon_0} - \frac{1}{\epsilon_0}\vec{\nabla}\cdot\vec{P}$$

It makes sense to define: $\vec{D} = (\epsilon_0 \vec{E} + \vec{P})$

$\vec{D}$ is called the electric displacement; it obeys

$$\vec{\nabla}\cdot\vec{D} = \rho_f \quad \text{or} \quad \int d^2x\,\hat{n}\cdot\vec{D} = Q_f$$

So the flux of $\vec{D}$ gives the free charge, just as the flux of $\vec{E}$ gives the total charge — free plus bound.

In a linear medium, $\vec{P} = \epsilon_0 \chi_e \vec{E}$. Then $\vec{D}$ is also linearly related to $\vec{E}$:

$$\vec{D} = \epsilon_0 \left(\vec{E} + \chi_e \vec{E}\right) = \epsilon_0 (1+\chi_e)\,\vec{E}$$

Write:

$$\epsilon = \epsilon_0(1+\chi_e) \quad \text{"permittivity"}$$

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = \text{"dielectric constant"}$$

$$= (1 + \chi_e)$$

in our crude model of atomic polarizability

$$\epsilon_0 \chi_e = \frac{n\alpha}{1 \text{ [redacted] } \frac{n\alpha}{3\epsilon_0}}$$

$$\chi_e = \frac{n\alpha/\epsilon_0}{1 \text{ [redacted] } n\alpha/3\epsilon_0} \qquad \epsilon_r = \left(\frac{1 \text{ [redacted] } \frac{2}{3}\frac{n\alpha}{\epsilon_0}}{1 \text{ [redacted] } \frac{1}{3}\frac{n\alpha}{\epsilon_0}}\right)$$

This last relation is often written as a formula for α in terms of ϵ_r:

$$\alpha = \frac{3\epsilon_0}{n}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$$

This is the usual form of the Clausius-Mossotti relation.

If we use our earlier result for α (p.9)

$$\alpha \sim 4\pi\epsilon_0 \left(1\,\mathring{A}\right)^3$$

$$\frac{n\alpha}{3\epsilon_0} \sim \left(\frac{4\pi}{3}a^3\right)\cdot n \sim (\text{atomic volume})\cdot(\text{density})$$

$$\sim 1$$

so we expect

$\epsilon_r > 1$ but ϵ_r of order 1 for a solid material

for some typical materials

$\epsilon_r = 1 + 5\times10^{-4}$ (air)

1.7 (glass)

5.9 (salt NaCl)

80.1 (water)

Let's recap: for a polarizable medium, we defined

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

then $\vec{D}, \vec{E}$ obey the electrostatic equations

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\times\vec{E} = 0$$

These equations are exact consequences of the fundamental electrostatic field equations.

If we further assume a linear relation between $\vec{P}$ and $\vec{E}$

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

then it follows that

$$\vec{D} = \epsilon \vec{E}$$

for some constant ϵ characteristic of the medium. This is a phenomenological relation that is valid only when $\vec{E}$ is small enough. However, this is typically true, and so the linear relation of $\vec{D}$ and $\vec{E}$ is a very useful approximation for solving practical problems.