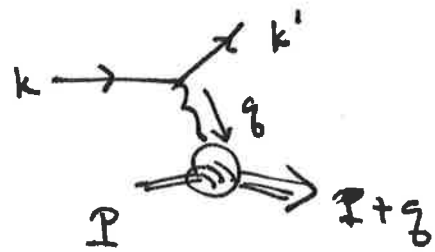
The Parton Model

In the previous lecture, we discuessed reactions with e^+e^- in the initial state. At the LHC, we have proton-proton collisions. The proton is of course a bound state of quarks and gluons whose structure is determined by nonperturbative interactions. We need a formalism for computing cross sections in reactions of this type.

A useful description of the proton structure is easiest to derive and motivate by thinking about *ep* scattering. I will begin by setting up the kinematic description for this process. We will be most interested in the limit of high energies and large momentum transfers, so i will simplify the formulae by ignoring the electron and proton masses with respect to the various kinematic invariants.

To leading order in electromagnetism, the *ep* scattering amplitude has the structure



The momentum transfer from the electron to the hadronic system is

$$q = k - k'$$

This momentum is spacelike, so we may write

$$q^2 = -Q^2 \qquad Q^2 > 0$$

The center of mass energy is

$$s = 2k \cdot P$$

The mass of the final hadronic system is

$$W^2 = (P+q)^2 = 2P\cdot q - Q^2$$

In the fixed target lab frame in which the proton is at rest, the fraction of the electron's energy that is transfered to the hadronic system is

$$y = \frac{q^0}{k^0} = \frac{2q\cdot P}{2k\cdot P} = \frac{2q\cdot P}{s}$$

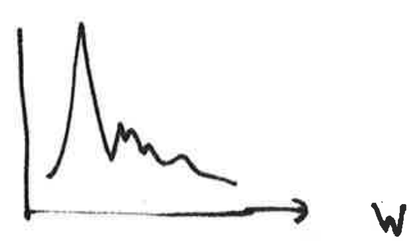
Finally, define

$$x = \frac{Q^2}{2q\cdot P}$$

so that

$$Q^2 = xys \qquad\qquad 0 \le x, y \le 1$$

It is interesting to look at the cross section data as a function of W for increasing values of Q. For low Q, we see the spectrum of baryon resonances



As Q increases, the resonances merge together into a broad structure

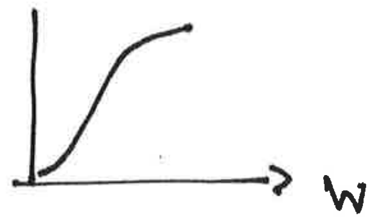


Figure 1 shows the data from the SLAC-MIT deep inelastic scattering experiment of the 1960's.

In the Standard Model, the proton is a composite state, a bound state of quarks and gluons. The strong interactions which bind them are described by QCD, an asymptotically free gauge theory. In this theory, large momentum transfer interactions are described by perturbation theory and are suppressed by powers of the coupling constant α_s. This suggests that we model the proton as a collection of constituents, called "partons", that share the momentum of the proton. These partons exchange only small momenta with one another within the proton wavefunction. If the proton is highly boosted, the partons move approximated collinearly, with momentum related to the total proton momentum P by

$$p = \xi P \qquad\qquad 0 < \xi < 1$$

The parameter ξ is called longitudinal fraction. The distribution of partons in this parameter is determined by nonperturbative physics. The probability to find a parton at fraction ξ can be parametrized by a "parton distribution function" or pdf,

$$f_i(\xi)\, d\xi$$

The pdfs for the various species of parton are properties of the proton. Thus, if we can determine them by relating them to some cross section, we know them once and for all, and we can use them to compute other cross sections involving the proton in the initial state.

In this model, we would compute the cross section for *ep* scattering as a collection of scattering processes for the electron with the various partons. The leading term comes from elastic scattering of the electron from the partons with electric charge – the quarks and antiquarks. Then, in the parton model, we have the diagram

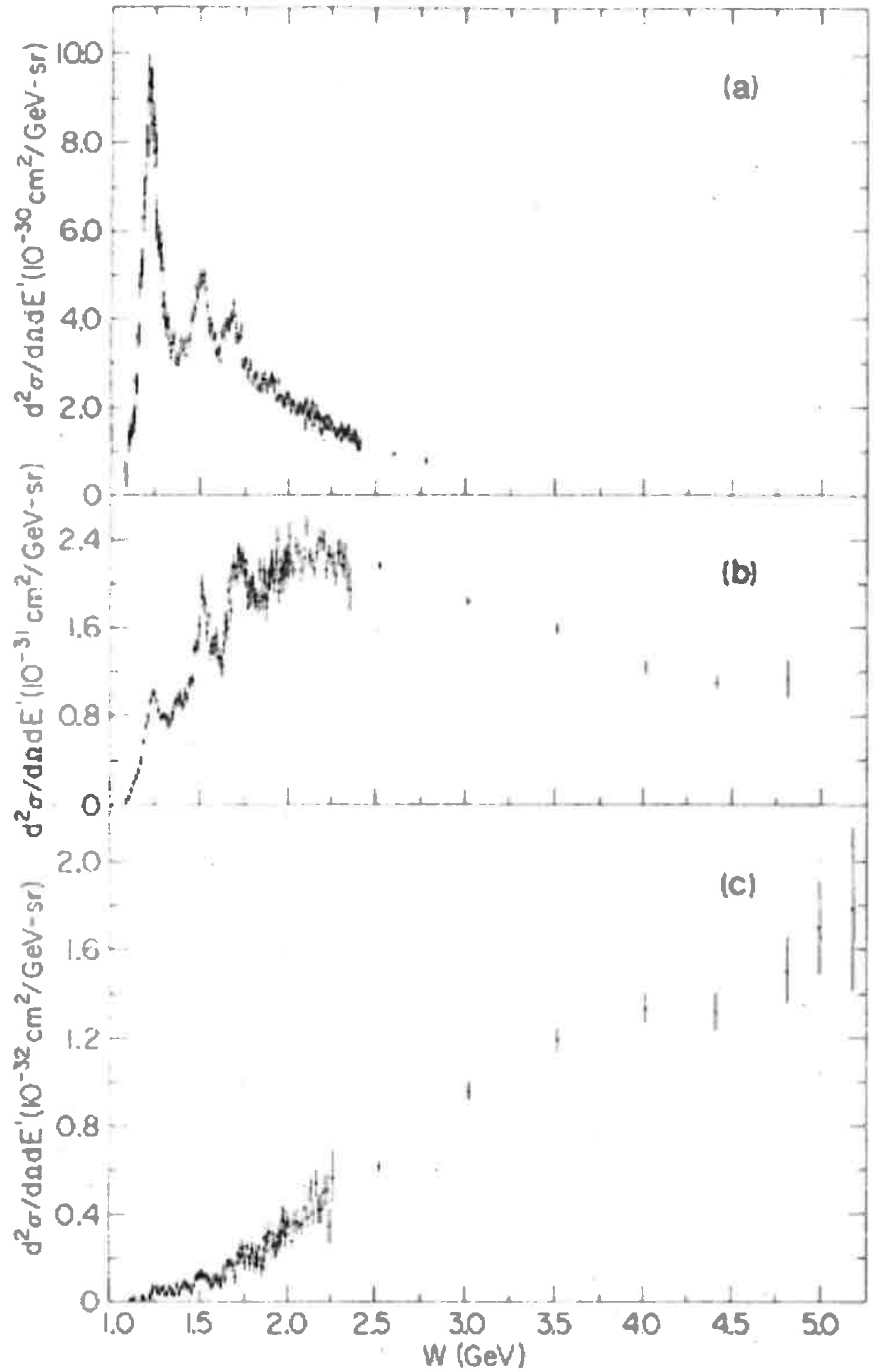
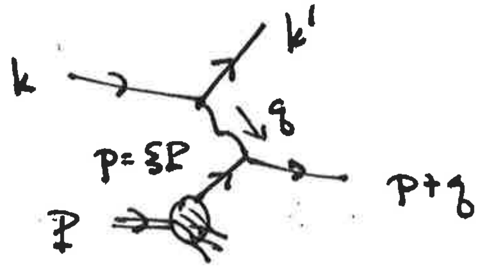


Fig. 1 $\bar{e}p$ · cross sections at increasing values of Q^2 as a function of W, from the SLAC-MIT experiment.
E.D. Bloom et al Phys. Rev. Lett. 23, 930 (1969)



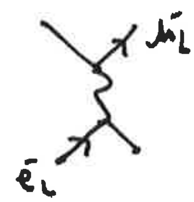
and the expression for the cross section

$$\sigma(e^- p) = \int d\xi \sum_i f_i(\xi) \int d\cos\Theta \, \frac{d\sigma}{d\cos\Theta} \left(e^- + p_i \to e^- p_i \right)$$

The functions $f_i(x)$ are the pdfs. The index i runs over quarks, index by flavor f and antiquarks, indexed by flavor $\overline{f}$.

We can extract the expression for the cross section in this equation from the analysis of the previous lecture. There, we derived

$$\frac{d\sigma}{d\cos\Theta} \left(e_L^- e_R^+ \to \mu_L^- \mu_R^+ \right) = \frac{\pi\alpha^2}{2s} (1+\cos\Theta)^2 .$$



For a 2-body scattering process with massless particles of momentum p, the kinematic invariance evaluate in the center of mass system to

$$\begin{aligned} s &= 4p^2 \\ t &= -2p^2(1-\cos\Theta) \\ u &= -2p^2(1+\cos\Theta) \end{aligned}$$

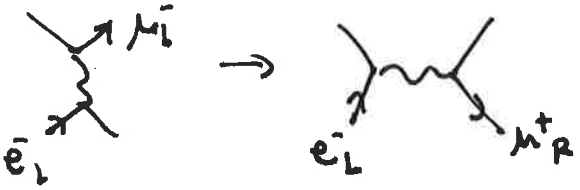
For this process

$$|\mathcal{M}|^2 = e^4 (1+\cos\Theta)^2 = 4e^4 \frac{u^2}{s^2}$$

We can then derive the results for t-channel photon exchange by crossing:

$$|\mathcal{M}|^2 = 4e^4 \frac{s^2}{t^2}$$

and



$$|\mathcal{M}|^2 = 4e^4 \frac{u^2}{t^2}$$

Then the scattering cross sections for polarized electrons with other polarized fermions are

$$\frac{d\sigma}{d\cos\theta}(e_L f_L) = \frac{2\pi\alpha^2}{s} \frac{s^2}{t^2}$$

$$\frac{d\sigma}{d\cos\theta}(e_L f_R) = \frac{2\pi\alpha^2}{s} \frac{u^2}{t^2}$$

In the forward direction, both cross sections take the form

$$\frac{d\sigma}{d\cos\theta} \sim \frac{2\pi\alpha^2}{s} \frac{1}{\left(\frac{1-\cos\theta}{2}\right)^2} \sim \frac{2\pi\alpha^2}{E_{cm}^2} \frac{1}{\sin^4\theta/2}$$

familiar from Coulomb scattering. The vanishing of the cross section for $e_L^-\mu_R^-$ scattering in the backward direction, where $u = 0$, is a consequence of angular momentum conservation

The spin-averaged cross section is

$$\frac{d\sigma}{d\cos\theta}(e^-\mu^-) = \frac{\pi\alpha^2}{s}\left(\frac{s^2+u^2}{t^2}\right)$$

For e^-q scattering, this formula is multiplied by the quark electric charge Q_f^2.

Using this equation, we have the parton model formula for ep scattering

$$\sigma(e^-p) \cong \int d\xi \sum_f Q_f^2 \left(f_f(\xi) + f_{\bar{f}}(\xi)\right) \int d\cos\theta \ \frac{\pi\alpha^2}{\hat{s}} \ \frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}$$

where f runs over the various quark flavors. In this formula and below, I will use the notation $\hat{s}$, $\hat{t}$, $\hat{u}$ to denote the kinematic invariants of a parton reaction.

To interpret this formula, we must relate the variables in it to measurable parameters of an ep scattering process. First,

$$\hat{s} = 2p\cdot k = \xi \cdot 2P\cdot k = \xi s$$

$$\hat{u} = 2p\cdot k' = \xi \cdot 2P\cdot(k-q) = \xi(1-y)s$$

The momentum transfer is

$$\hat{t} = q^2 = -Q^2 \qquad d\hat{t} = -\tfrac{1}{2}\hat{s}\, d\cos\theta$$

With these identifications,

$$\sigma(e^-p) \cong \int d\xi\, dQ^2 \sum_f Q_f^2 \left[f_f(\xi) + f_{\bar{f}}(\xi)\right] \frac{2\pi\alpha^2}{(Q^2)^2}\left[1+(1-y)^2\right]$$

The final step in the derivation uses an important kinematic observation due to Feynman: In elastic eq scattering with massless partons, the zero mass condition for the final state parton is just

$$0 = (p+q)^2 = 2\xi P\cdot q - Q^2$$

or

$$\xi = \frac{Q^2}{2P\cdot q} = x$$

In each ep event, we can measure x, and this value gives the momentum fraction of the parton that was struck by the electron. Then the cross section formula becomes

$$\frac{d\sigma}{dx\,dQ^2}(ep \to eX) = \sum_f Q_f^2 \left[f_f(x) + f_{\bar{f}}(x)\right] \frac{2\pi\alpha^2}{(Q^2)^2}\left[1+(1-y)^2\right]$$

The derivation applies, at least as a first approximation, when the momentum transfer from the electron is much larger than the internal momentum transfers within the proton wavefunction.

The final result can be written

$$\frac{d\sigma}{dx\,dy} = F_2(x)\;\frac{2\pi\alpha^2 s}{(Q^2)^2}\left[1+(1-y)^2\right]$$

where

$$F_2(x) = \sum_f Q_f^2\; x\left[f_f(x) + f_{\bar{f}}(x)\right]$$

This idea that the ep scattering cross section factorizes in this way between x and y is quite amazing. Bjorken predicted this factorization, and it is now called "Bjorken scaling". The prediction is that, when the measured cross section is divided by the simple QED expression on the right, the resulting value of $F_2(x)$ such be a universal function of x. This observation worked extremely well for the data set of the SLAC-MIT experiment and brought order to the data set shown earlier. Figure 2 shows the SLAC-MIT data points with $Q > 1$ GeV expressed as values of $F_2(x)$.

The pdfs $f_f(x)$ that enter this analysis represent the proton wavefunction. To give the proton the correct quantum numbers, we must have

$$\int_0^1 dx\, [f_u(x) - f_{\bar{u}}(x)] = 2$$

$$\int_0^1 dx\, [f_d(x) - f_{\bar{d}}(x)] = 1$$

and, for all other quarks,

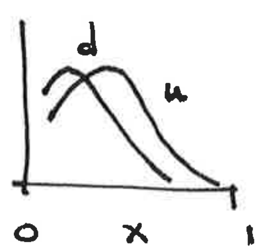
$$\int_0^1 dx\, [f_q(x) - f_{\bar{q}}(x)] = 0$$

The full set of partons should carry the full momentum of the proton, so that

$$\int_0^1 dx\, x\, \Big\{ \sum_f \big(f_f(x) + f_{\bar{f}}(x)\big) + f_g(x) \Big\} = 1$$

where $f_g(x)$ is the parton distribution of the gluon. It is useful to divide the u and d parton distribution into a "valence" part that satisfies the above sum rules with quarks only and a "sea" part with an equal number of quarks and antiquarks.

I would now like to make some rough sketches of parton distributions that satisfy these constraints. To discuss the precise values of parton distributions, we will need some additional physics that I will present in the next lecture. For the moment, here are some pictures of the pdfs based on intuition: The valence quark distributions should have the form



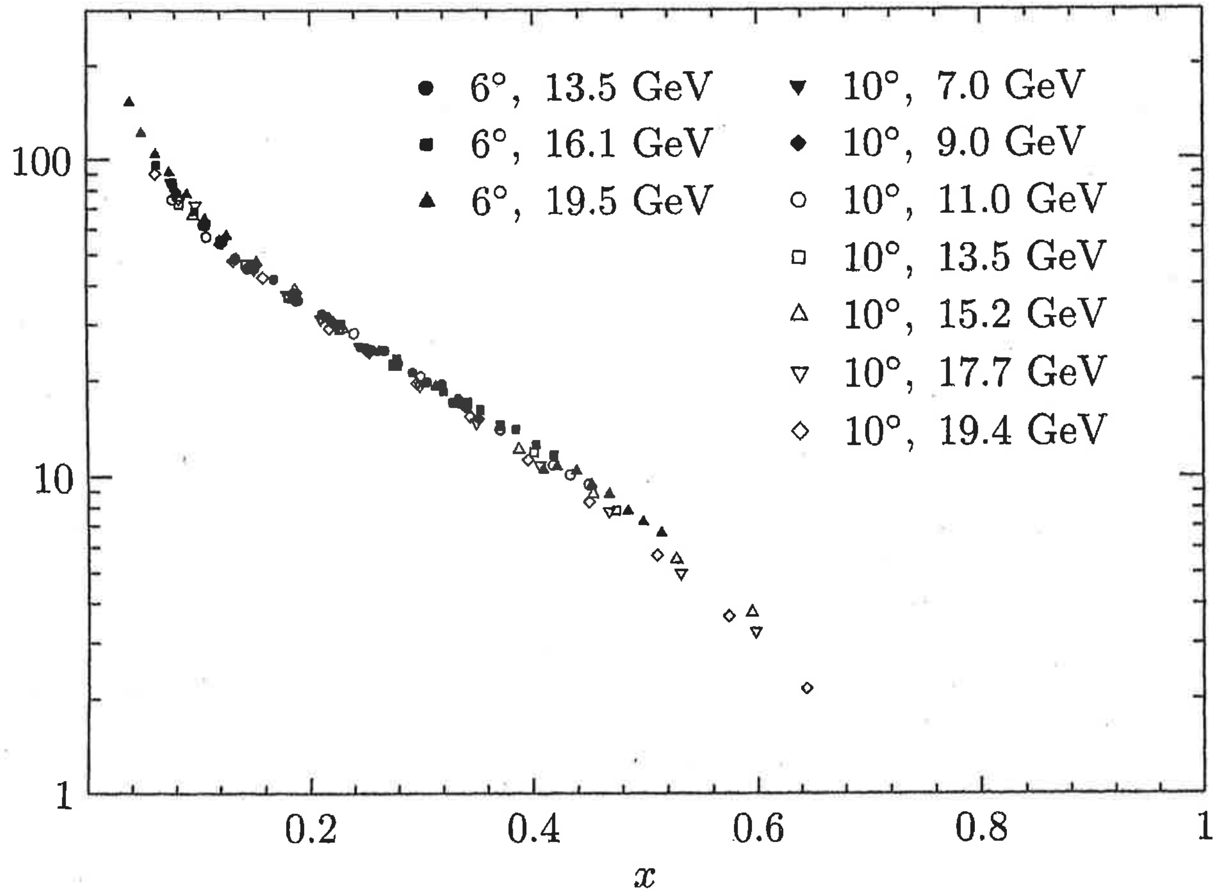
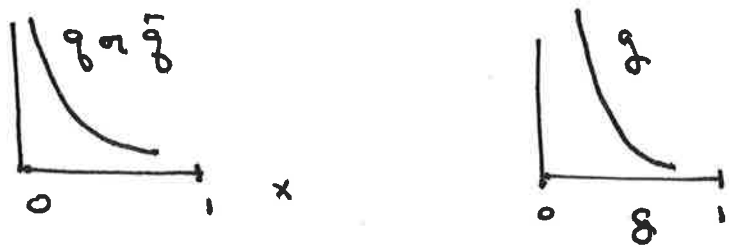


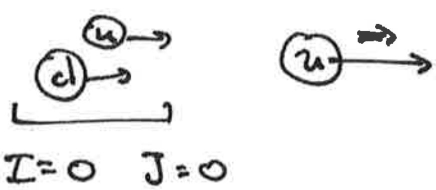
Fig. 2 Values of $F_2(x)$ measured by the SLAC-MIT experiment at a variety of values of Q^2.

from Peskin + Schroeder, "An Introduction to Quantum Field Theory", ch. 14.

Sea quarks, antiquarks, and gluons should be present only a small values of the momentum fraction,

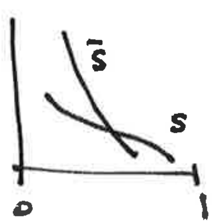


The u and d valence distributions need not be equal. In fact, if one quark is to carry most of the momentum of the proton, the easiest way to arrange this is in a configuration



so we might expect that the u quarks are generally a higher x than the d quarks. Similarly, the simplest way to have antiquarks in the proton is through the quantum fluctuation $p \leftrightarrow n + \pi^+$.

So the $\overline{d}$ antiquarks should be an higher x than the $\overline{u}$'s. Similarly, the easiest way to have strange quarks in the proton is through $p \leftrightarrow \Lambda^0 K^+$, so the strange quark s and $\overline{s}$ distributions can be different



Data from electron scattering supports all three of these suggestions.

The normalization of $F_2(x)$ gives the fraction of the proton's momentum that is carried by quarks and antiquarks. In fact, this normalization is lower than one might expect, indicating that about 50% of the protoin's momentum is carried by gluons.

Armed with this description of proton structure, we can write formulae for cross sections in proton-proton scattering. The simplest case is for the Drell-Yan process, $q\overline{q}$ annihilation to a virtual photon or electroweak boson, visible eventually as a lepton pair. For concreteness, I will write the reaction as $pp \to \mu^+\mu^- + X$. The elementary parton-parton cross section that we need for this processes is again the one derived in the previous lecture, with

$$\frac{d\sigma}{d\cos\theta}(q_f\bar{q}_f \to \mu^+\mu^-) = \frac{Q_f^2}{3}\frac{\pi\alpha^2}{2\hat{s}}(1+\cos^2\hat{\theta})$$

for a pure QED process, and the more complicated formula derived in the previous lecture to include γ-Z interference. The factor 1/3 comes from color: the parton wavefunctions contain quarks of all three colors, and only 1/3 of the time does the quark color and antiquark anticolor match.

The cross section for the Drell-Yan process is, then

$$\sigma(pp \to \mu^+\mu^- + X) = \int_0^1 dx_1 \int_0^1 dx_2 \sum_f \left[f_f(x_1) f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f})\right] \frac{d\sigma}{d\cos\theta}(q\bar{q} \to \mu^+\mu^-)$$

A priori, we do not know the momentum fractions carried by the quark or antiquark. However, it is a very nice property of this and other 2-body reactions that we can extract this information by observing the final state. The final 2-muon system has a total momentum vector

$$p(\mu^+) + p(\mu^-) = (\mathcal{E}, 0, 0, \mathcal{P})$$

In the parton model, and as observed, the muon pairs dominantly have small transverse momentum relative to the beam axis; I will ignore this transverse momentum in the rest of this argument.

Given that the initial-state quark and antiquark are massless particles moving in the beam direction, we need to decompose this total monentum vector as

$$x_1 E\,(1\,0\,0\,1) + x_2 E\,(1\,0\,0\,-1)$$

This is simple, but we can make it even simpler by introducing a new variable y, called "rapidity". In relativistic kinematics

$$\mathcal{E}^2 = \mathcal{P}^2 + M^2$$

so we can parametrize a 4-vector as

$$\mathcal{E} = M \cosh y \qquad \mathcal{P} = M \sinh y$$

The rapidity is a very convenient parameter for collider physics. Under boosts in the longitudinal direction, it shifts in a simple way

$$(\mathcal{E}, \mathcal{P}) \to (\gamma \mathcal{E} + \gamma\beta \mathcal{P},\ \gamma \mathcal{P} + \gamma\beta \mathcal{E})$$

Since

$$\gamma^2 - (\gamma\beta)^2 = \gamma^2(1-\beta^2) = 1$$

we can write

$$\gamma = \cosh \alpha \qquad \gamma\beta = \sinh \alpha$$

and then

$$(\mathcal{E}, P) \rightarrow M(\cosh\alpha\cosh y + \sinh\alpha\sinh y, \cosh\alpha\sinh y + \sinh\alpha\cosh y)$$

$$= M(\cosh(y+\alpha), \sinh(y+\alpha))$$

Thus, a longitudinal boost corresponds to a simple shift of the rapidity. in proton collider physics, wew do not know a priori the parton-parton center of mass system for each event. Using rapidity as a variable makes the description of these events equivalent for all longitudinally boosted frames.

For a particle with transverse momentum

$$P = (E, p_T, p_{\parallel})$$

we have

$$P^2 = M^2 = E^2 + p_{\parallel}^2 + (p_T^2 + m^2)$$

It is convenient to define the "transverse mass" as

$$m_T^2 = (p_T^2 + M^2)$$

Then

$$\mathcal{E} = m_T \cosh y \qquad p_{\parallel} = m_T \sinh y$$

and the description again transforms under longitudinal boosts as a simple shift of the rapidity. Transverse mass is a very important variable in many LHC analyses.

We can now return to the problem posed above. We had the total momentum vector

$$p(\mu^+\mu^-) = M(\cosh y, 0, 0, \sinh y)$$

Then obviously

$$p(\mu^+\mu^-) = M\frac{e^{y}}{2}(1\,0\,0\,1) + M\frac{e^{-y}}{2}(1\,0\,0\,-1)$$

We can identify

$$x_1 = \frac{M}{2E}e^{y} \qquad x_2 = \frac{M}{2E}e^{-y}$$

with

$$2E = \sqrt{s}$$

The Jacobian for the change of variables from $dx_1 dx_2$ to $dM dy$ is

$$\frac{\partial(x_1, x_2)}{\partial(M y)} = \begin{vmatrix} \frac{1}{\sqrt{s}}e^{y} & \frac{1}{\sqrt{s}}e^{-y} \\ \frac{M}{\sqrt{s}}e^{y} & -\frac{M}{\sqrt{s}}e^{-y} \end{vmatrix} = \frac{2M}{s}$$

so the Drell-Yan cross section formula becomes

$$\sigma(pp \to \mu^+\mu^- + X) = \int dy\, dM\ \frac{2M}{s} \sum_f \left[f_f(x_1) f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f}) \right] \cdot \int d\cos\hat{\theta}\ \frac{d\sigma}{d\cos\hat{\theta}}$$

or

$$\frac{d\sigma}{dy\,dM^2\,d\cos\hat{\theta}} = \sum_f \left[f_f(x_1) f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f})\right] \frac{1}{s} \frac{d\hat{\sigma}}{d\cos\hat{\theta}}$$

It is useful to take one more step. Since

$$\hat{s} = 2x_1 P_1 \cdot x_2 P_2 = x_1 x_2 s$$

we can write, finally,

$$\frac{d\sigma}{dy\,dM^2\,d\cos\hat{\theta}} = \sum_f \left[x_1 f_f(x_1)\, x_2 f_{\bar{f}}(x_2) + (f \leftrightarrow \bar{f})\right] \frac{1}{\hat{s}} \frac{d\hat{\sigma}}{d\cos\hat{\theta}}$$

where x_1, x_2 are determined from the event kinematics as described above. The derivation we have given for the Drell-Yan process actually applies to any $2 \to 2$ parton scattering process.

Figures 3 and 4 show the Drell-Yan cross section $d\sigma/dM$ as measured by the CMS collaboration at 8 TeV, first as a function of M and then as a function of rapidity. The mass dependence is very similar to the dependence on $\sqrt{s}$ for the e^+e^- cross sections studied in the previous lecture. Notice how the rapidity distribution contracts as M increases. From the analysis above,

$$M^2 = x_1 x_2 s$$

But, for high M, both x_1 and x_2 must be large, so it is more difficult to make their ratio large.

In the vicinity of the resonance, we can consider Z, or W, production as a $2 \to 1$ process. The matrix element for W boson production is

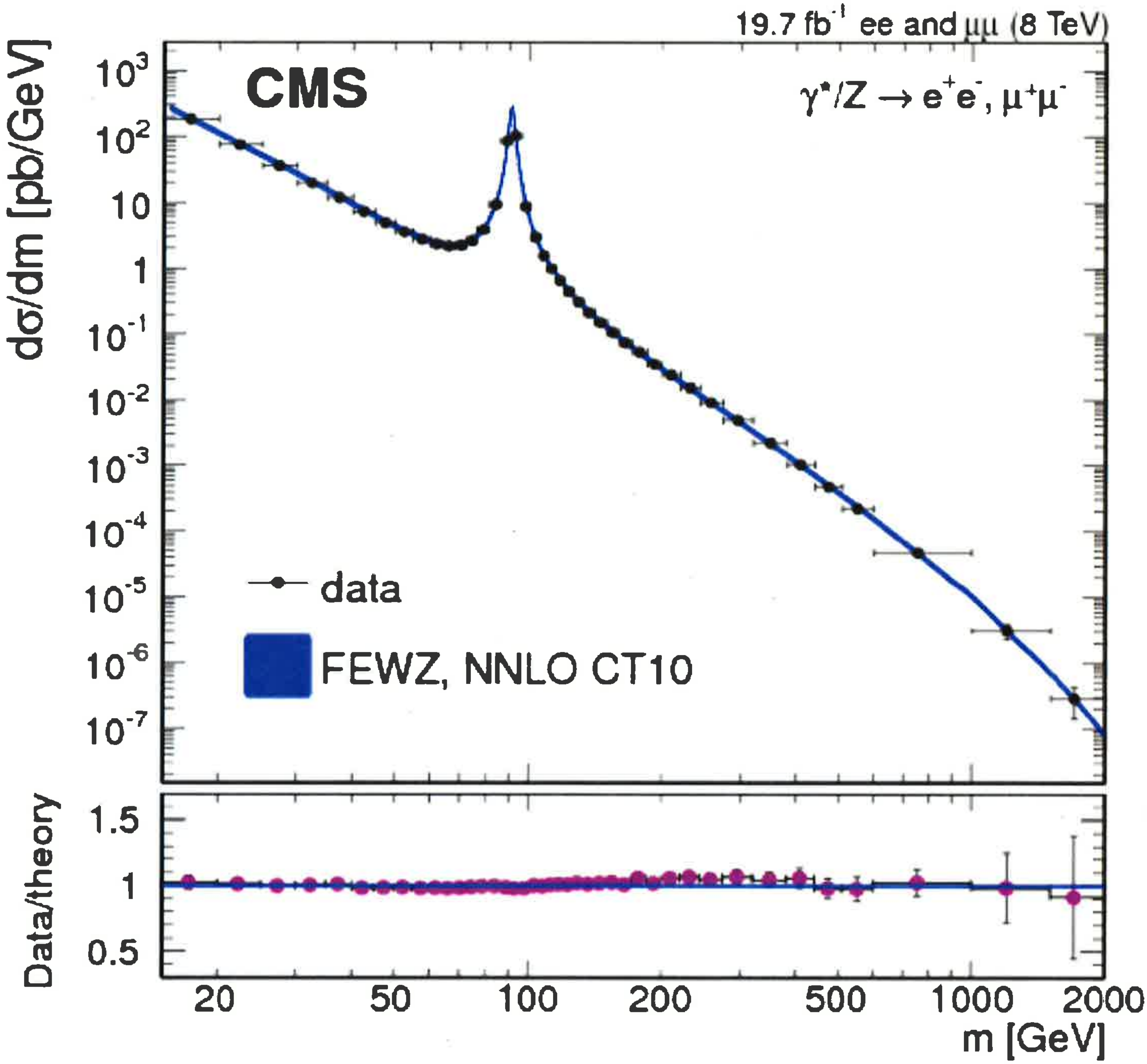


Fig. 3 The Drell-Yan cross section as a function of $m(\mu^+\mu^-)$ measured by the CMS collaboration, arXiv:1412.1115.

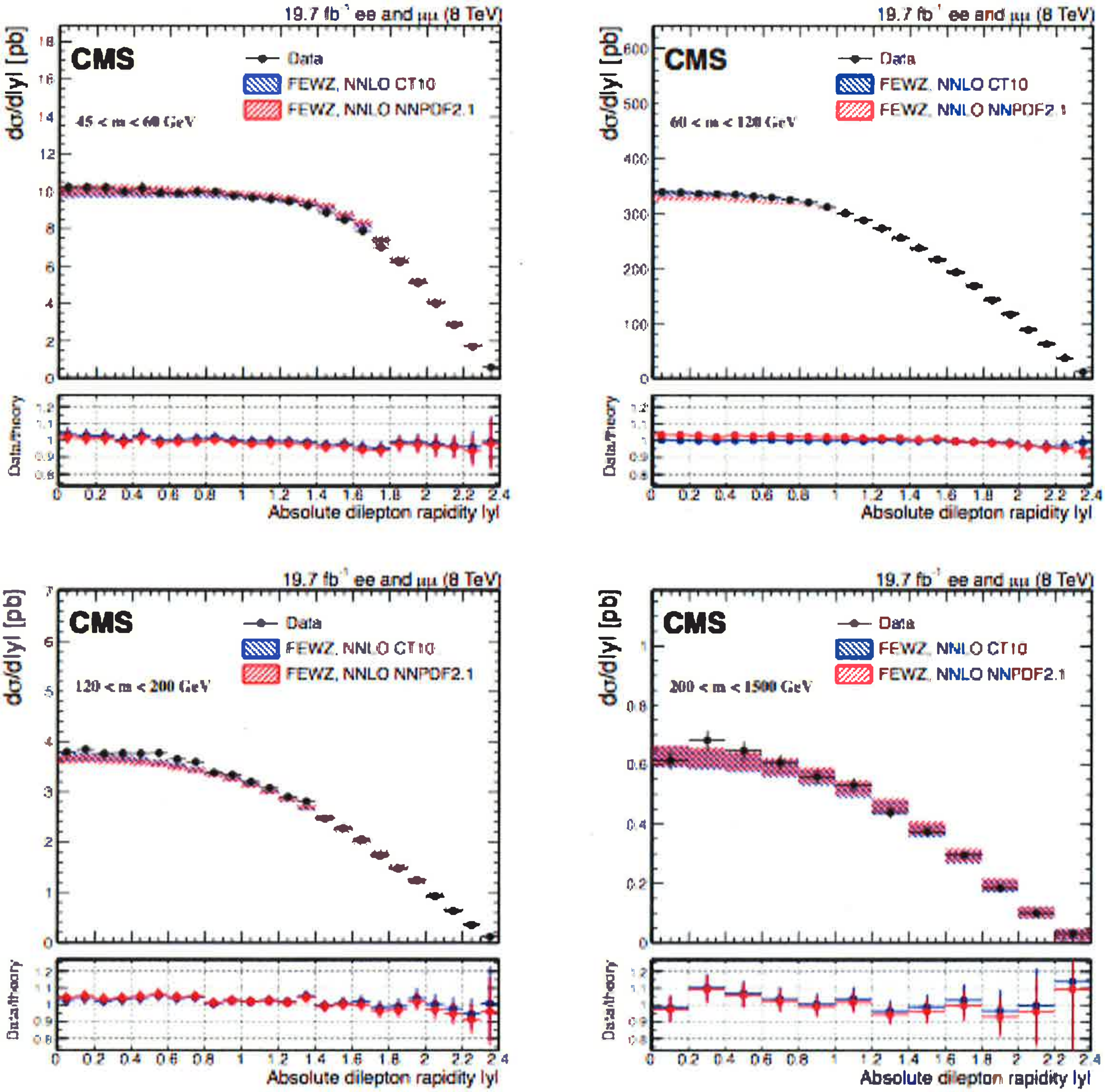


Fig. 4 The Drell-Yan cross section as a function of Y, in different regions of $m(\mu^+\mu^-)$, as measured by the CMS collaboration, arXiv: 1412.1115

$$\mathcal{M}(u\bar{d}\to W^+) = \frac{ig}{\sqrt{2}} \cdot \sqrt{2}\, 2E \quad \varepsilon^*(W)\cdot\varepsilon_L(p)$$

Then

$$|\mathcal{M}|^2 = g^2 m_W^2 \quad |\varepsilon(W)\cdot\varepsilon_L(p)|^2$$

The cross section is computed from

$$\sigma(q\bar{q}\to W) = \frac{1}{2\hat{s}} \int d\Pi_1 \, |\mathcal{M}|^2 \cdot \frac{1}{3}\cdot\frac{1}{4}$$

where $\frac{1}{3}$ is the color factor, $\frac{1}{4}$ comes from the spin average, and

$$\int d\Pi_1 = \int \frac{d^3 p_W}{(2\pi)^3 2E_W} (2\pi)^4 \delta^{(4)}(p_1+p_2 \to p_W)$$

$$= \int \frac{d^4 p_W}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(\;) \cdot 2\pi\, \delta(p_W^2 - m_W^2)$$

Assembling the pieces,

$$\sigma(u\bar{d}\to W^+) = \frac{1}{12}\cdot\frac{1}{2m_W^2}\, 2\pi\delta(\hat{s}-m_W^2)\cdot g^2 m_W^2$$

or

$$\sigma(u\bar{d}\to W^+) = \frac{\pi^2 \alpha_w}{3}\,\delta(\hat{s}-m_W^2)$$

This leads to

$$\frac{d\sigma}{dy}(pp \to W^+) = \frac{\pi \alpha_w}{3 m_W^2} \left[x_1 f_u(x_1)\, x_2 f_{\bar{d}}(x_2) + (1 \leftrightarrow 2) \right]$$

The corresponding formula for the Z is

$$\sigma(u\bar{u} \to Z^0) = \frac{2\pi^2 \alpha_w}{3 c_w^2} \left(Q_{ZuL}^2 + Q_{ZuR}^2 \right) \delta(\hat{s} - m_Z^2)$$

and similarly for other $q\overline{q}$ pairs.

The W production cross sections have some interesting asymmetries in rapidity. Previous generations of hadron collider experiments were done with proton-antiproton collisions. Most recently, $p\overline{p}$ collisions were studied at a center of mass energy of 2 TeV at the Tevatron collider at Fermilab. A proton has valence quarks, and an antiproton has valence antiquarks, and it is the collisions of these valence partons that dominate the vector boson production. Since the u quarks in the proton are harder than the d quarks, the expectation is that W^+ production, from $u\overline{d}$, is shifted to the proton direction while W^- production, from $d\overline{u}$, is shifted to the antiproton direction. This effect was actually observed at the Tevatron. Figure 5 shows the data from the CDF experiment.

At the LHC, we have only protons, and so vector boson production comes from annihilation of sea antiquarks on valence or sea quarks. The experimental setup is symmetric, and so the boson production must be symmetric between y and $-y$. However, if the valence u quarks are harder than the valence d quarks, the production of W^+ will extend out to higher values of $|y|$ than that of W^-. This effect is observed at the LHC. Figure 6 shows the lepton charge asymmetry from $W^+ \to \ell^+\nu/W^- \to \ell^-\overline{\nu}$, as measured by the CMS experiment at 7 TeV.

The same formula that we have just derived also applied to quark-antiquark scattering, and, more generally, to all parton-parton scattering processes in QCD. From our experience with $Z \to q\overline{q}$ reactions, we should expect that the scattered quarks and gluons are manifested as jets with opposite momentum transverse to the beam direction. The cross section formula is

This leads to

$$\frac{d\sigma}{dy}(pp \to W^+) = \frac{\pi \alpha_w}{3 m_W^2}\left[x_1 f_u(x_1) x_2 f_{\bar{d}}(x_2) + \cdots\right]$$

The corresponding formula for the Z is

$$\frac{d\sigma}{dy}(pp \to Z^0) = \frac{2\pi^2 \alpha_w}{3 c_w^2 m_Z^2}\left(Q_{Zf_L}^2 + Q_{Zf_R}^2\right)$$

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The same formula that we have just derived also applied to quark-antiquark scattering, and, more generally, to all parton-parton scattering processes in QCD. From our experience with $Z \to q\overline{q}$ reactions, we should expect that the scattered quarks and gluons are manifested as jets with opposite momentum transverse to the beam direction. The cross section formula is

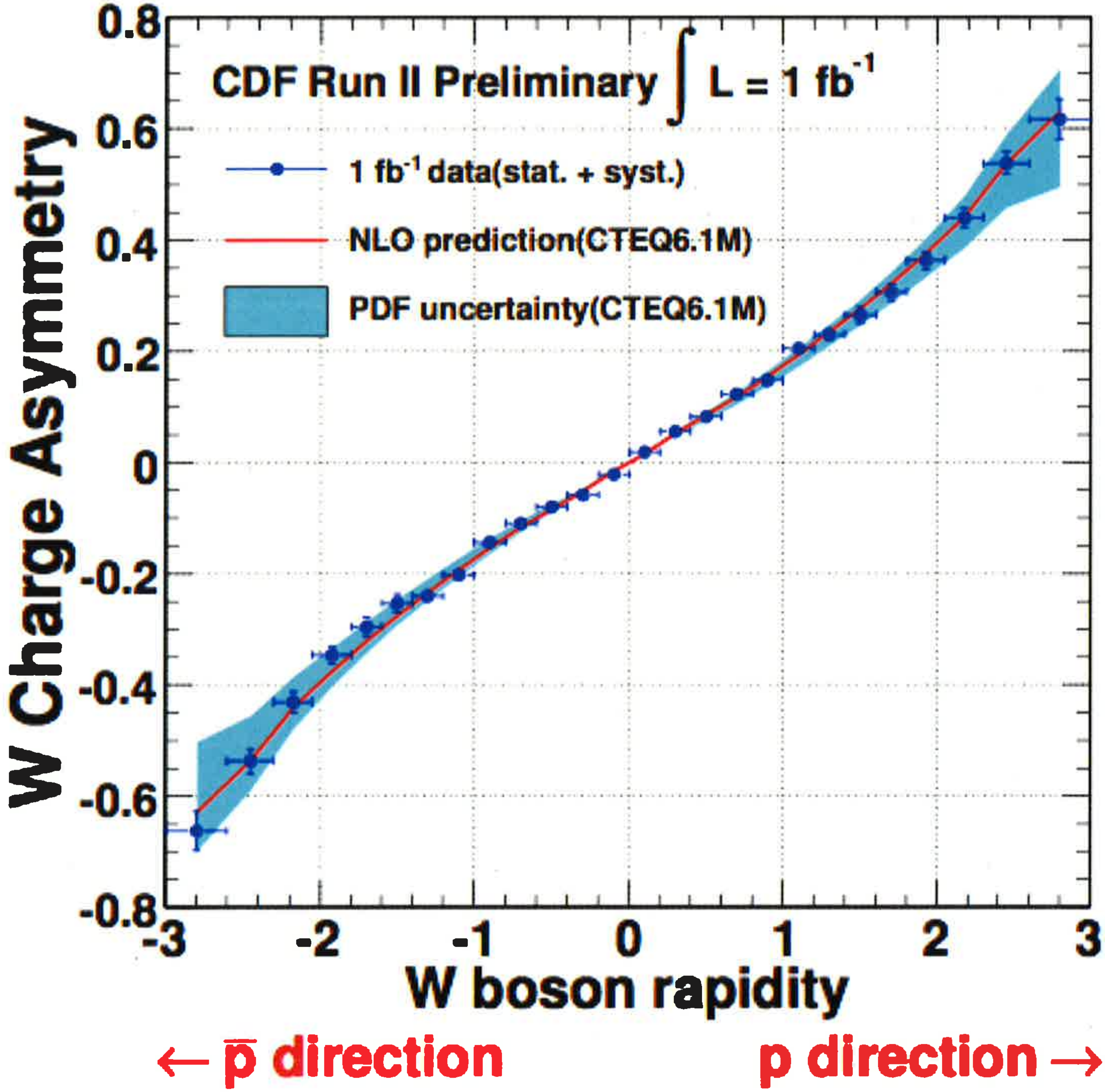

Fig. 5 W boson charge asymmetry measured in $p\bar{p}$ collisions by the CDF collaboration, public note 8942; see also Phys. Rev. Lett. 102, 181801 (2009)

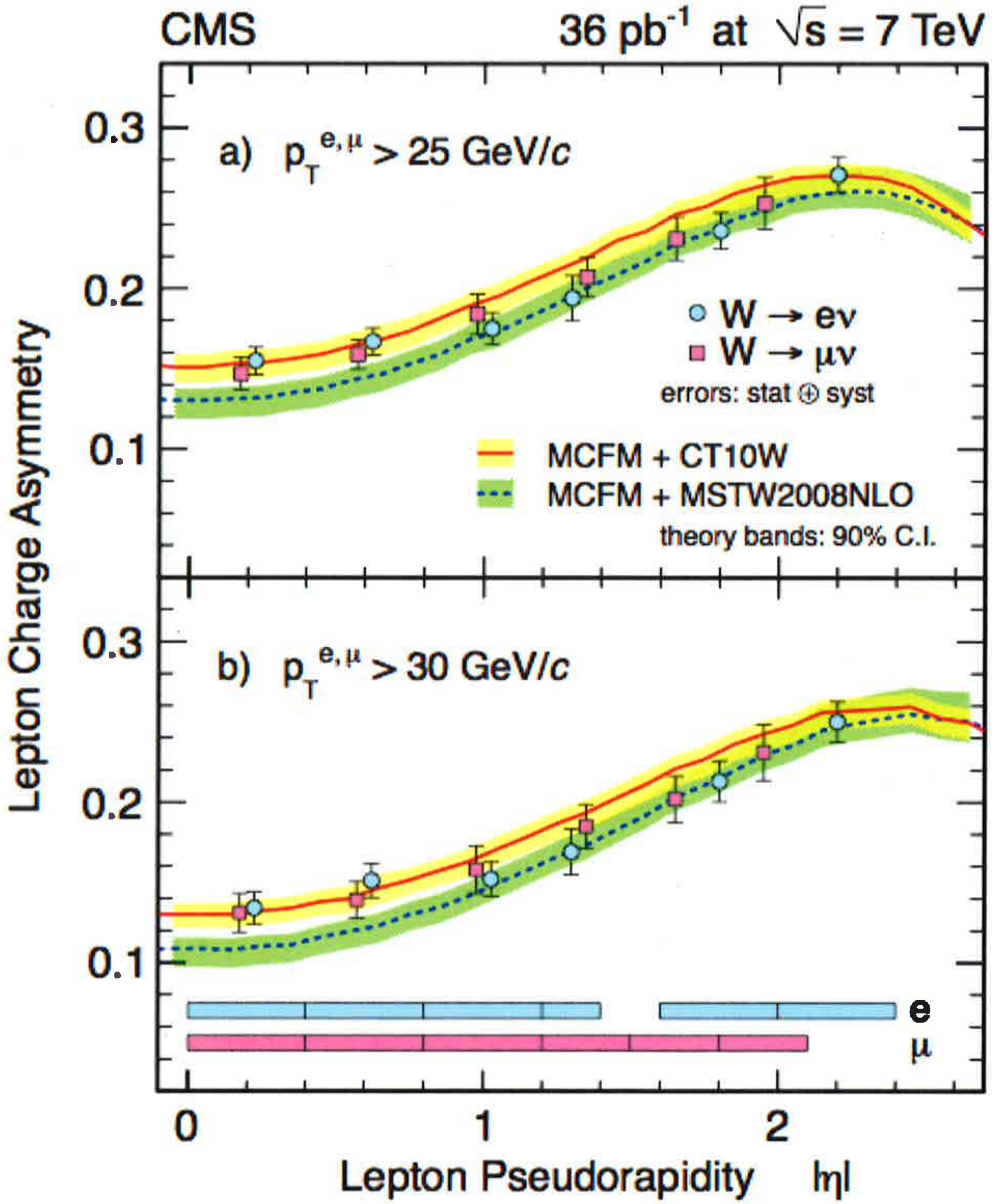

Fig. 6 Measurement of the lepton charge asymmetry vs. η for $pp \to W \to \ell\nu$ events, by the CMS collaboration, arXiv:1310.7291, JHEP 12, 030 (2013).

$$\frac{d\sigma}{dy\, d\hat{s}\, d\cos\hat{\theta}}(pp \to 2\text{ jets})$$

$$= \sum_{f_1 f_2} x_1 f_{f_1}(x_1)\, x_2 f_{f_2}(x_2)\ \frac{1}{s}\frac{d\sigma}{d\cos\theta}(1+2 \to 2\text{ jets})$$

I am sorry that there is no time in these lectures to compute the relevant QCD cross sections. You can find a relatively compact derivation in my lectures notes referenced in the Introduction, or an explanation of the method in Schwartz's QFT book. Here I will give a simple gloss on these formulae, from the following perspective: All elastic parton-parton cross sections must be dominated in the forward direction by Coulomb gluon exchange. Earlier in this lecture, I wrote the QED cross section for e^-q scattering,

$$\frac{d\sigma}{d\cos\theta}(e^-q \to e^-q) = \frac{\pi\alpha^2}{s} Q_f^2 \left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right)$$

to recover the result for qq scattering through QCD, and other similar processes, we just have to replace α by α_s and the electric charge Q_f^2 by the relevant QCD charge. For qq scattering, the required color factor is

$$\frac{1}{3}\cdot\frac{1}{3}\sum_{\text{color}} |t^a \cdot t^a|^2 = \frac{1}{9}\, \mathrm{tr}[t^a t^b]\, \mathrm{tr}[t^a t^b]$$

$$= \frac{1}{9}\cdot\frac{1}{2}\delta^{ab}\cdot\frac{1}{2}\delta^{ab} = 2/9$$

so

$$\frac{d\sigma}{d\cos\theta}(ud \to ud) = \frac{2\pi\alpha_s^2}{9\hat{s}}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right)$$

For scattering of identical quarks, there are some extra interference terms

$$\frac{d\sigma}{d\cos\theta}(uu \to uu) = \frac{2\pi\alpha_s^2}{9s}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2} + \frac{\hat{s}^2+\hat{t}^2}{\hat{u}^2} - \frac{2}{3}\frac{\hat{s}^2}{\hat{t}\hat{u}}\right)$$

For quark-gluon scattering, we can evaluate the forward scattering cross section by rescaling the squared QCD charge from the 3 to the 8 representation,

$$3:\ t^a t^a = \frac{4}{3} \quad \to \quad 8 \quad T^a T^a = 3$$

Then

$$\frac{d\sigma}{d\cos\theta}(ug) \sim \frac{\pi\alpha^2}{s}\frac{\hat{s}^2}{\hat{t}^2}$$

The complete formula is

$$\frac{d\sigma}{d\cos\theta}(ug \to ug) = \frac{\pi\alpha_s^2}{2s}\left[\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2} - \frac{\hat{u}}{\hat{s}} - \frac{\hat{s}}{\hat{u}}\right]$$

For gluon-gluon scattering, we need another factor of $\frac{9}{4}$,

$$\frac{d\sigma}{d\cos\theta}(gg \to gg) \sim \frac{9}{4}\frac{\pi\alpha^2}{s}\left(\frac{\hat{s}^2}{\hat{t}^2}\right)$$

The complete formula is

$$\frac{d\sigma}{d\cos\theta}(gg \to gg) = \frac{9}{4}\frac{\pi\alpha_s^2}{s}\left[3 - \frac{\hat{t}\hat{u}}{\hat{s}^2} - \frac{\hat{s}\hat{u}}{\hat{t}^2} - \frac{\hat{s}\hat{t}}{\hat{u}^2}\right]$$

The full set of formulae for $2 \to 2$ parton-parton scattering are given in Section 17.4 of my QFT textbook, or in the lecture notes referenced above.

Figure 7 shows the result of applying these formulae to compute the 2-jet production cross section at the LHC. Note the large importance of gluon-initiated processes. Valence quark-valence quark scattering is dominant only at the highest values of p_T or $\hat{s}$. I will discuss this figure further in lecture 4 of this series.

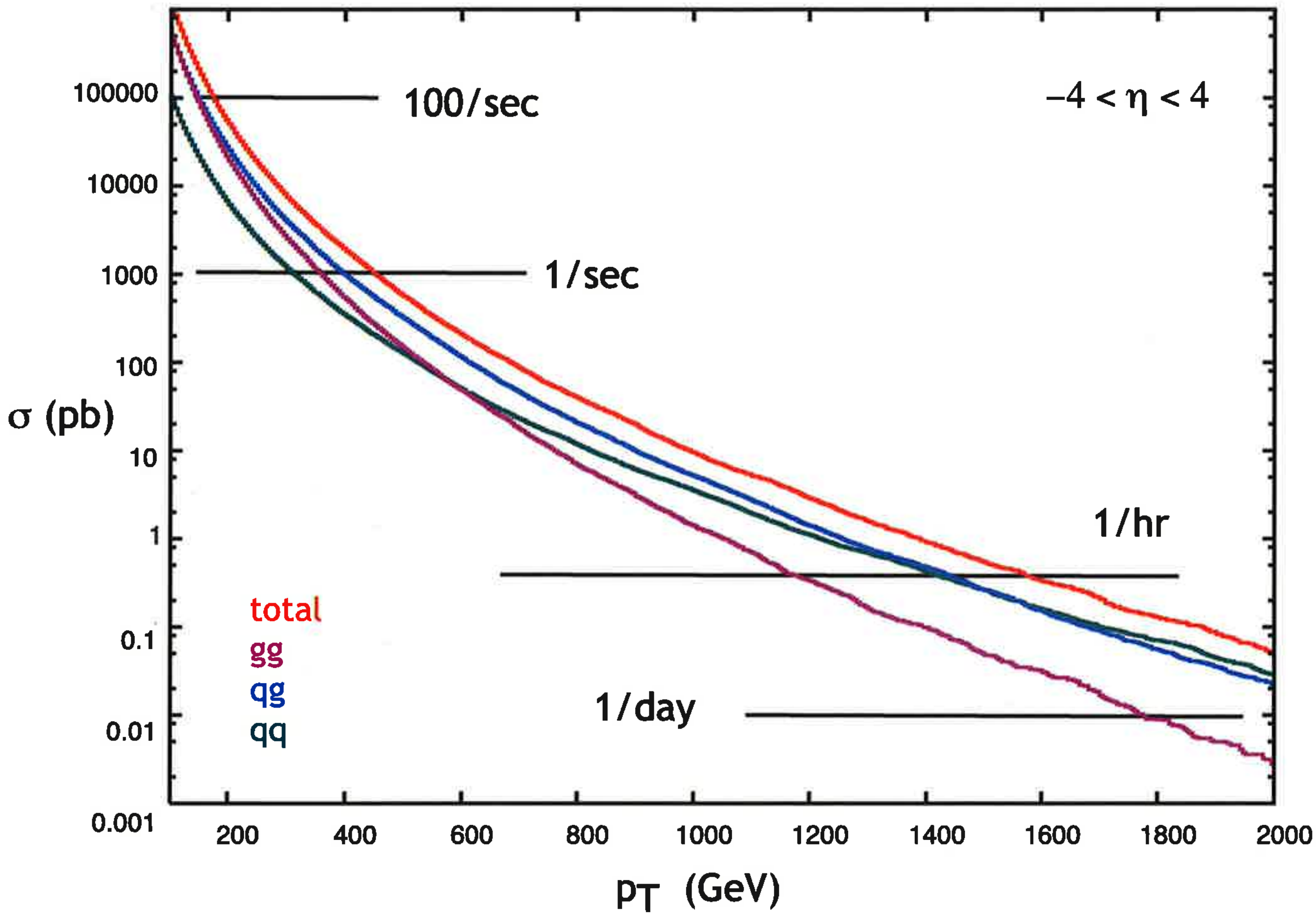

Fig. 7 Simple parton-model estimate of 2-jet cross section in pp collisions at 14 TeV, showing the contribution of gg, qg, and qq initial states. What is plotted is the integrated rate $\int_{p_T}^{\infty} dp_T \frac{d\sigma}{dp_T}$