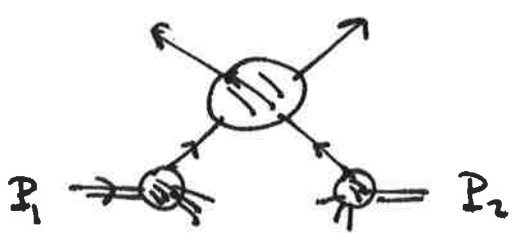
Altarelli-Parisi Evolution

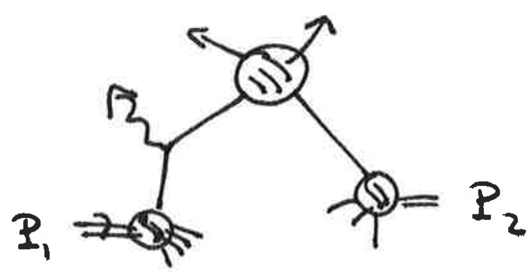
Before we discuss the physics of quarks and gluons more quantitatively, there is one more idea that we need to discuss. This is the evolution of quark and gluon distributions with scale. I will discuss this first for parton distributions, then generalize to reactions in which quarks and gluons appear in the final state.

In our computations with the parton model in the previous lecture, we assumed that all initial-state partons have zero transverse momentum. This is obviously an approximation. A quark can move to finite transverse momentum by several mechanisms, but the one that most effective for large transverse momentum is for the quark to emit a gluon and recoil. I will now work out the rate of this process in some detail.

The leading-order diagram for quark-quark scattering is



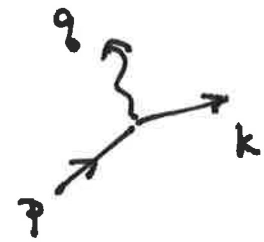
As explained in the previous lecture, it is a convolution of a hard-scattering cross section with two parton distributions. I will consider the hard process to have momentum transfer Q. A higher-order correction to this process is the emission of a gluon from the initial state



The gluon carries away transverse momentum $-p_T$ and leaves the quark with transverse momentum p_T. As long as $p_T \ll Q$, the expression for the hard-scattering cross section is not significantly effected. You might expect that this emission is suppressed by a power of α_s and by powers of $1/p_T$. However, it will turn out that the emission is enhanced at high energy, and that this emission is an order-1 effect. This happens because the gluon emission an leave the quark very close to its mass shell.

Write the initial quark momentum as

$$p = (E, 0, 0, E)$$



For $p_T \ll E$, the gluon momentum can be written

$$q = \left((1-z)E,\ -p_T,\ (1-z)E - \frac{p_T^2}{2(1-z)E} + ..\right)$$

The (off-shell) quark momentum is then

$$k = \left(zE,\ p_T,\ zE + \frac{p_T^2}{2(1-z)E} + ..\right)$$

I have parametrized these vectors in such a way that the quark retains a fraction z of its original momentum, while the gluon carries away the rest. The quark is off-shell by the amount

$$k^2 = (zE)^2 - p_T^2 - (zE)^2 - \frac{p_T^2\, zE}{(1-z)E}$$

$$= -p_T^2\left(1 + \frac{z}{1-z}\right) = -\frac{p_T^2}{(1-z)}$$

Note that this vanishes as $p_T \to 0$, and that the diagram is singular in this limit. This is called a "collinear singularity". This singularity is quite distinct from the "soft singularity" in QED when we emit a very soft photon. The collinear singularity is present only for massless particles emitting other massless particles. It appears because a lightlike vector can split into two parallel lightlike vectors that are arbitrary fractions of the original one.

By this logic, a massless particle can give up a large fraction of its longitudinal momentum in a single interaction with high probability. Such a process is called a "collinear splitting".

The amplitude for a collider splitting of a quark into a quark and gluon is

$$i\mathcal{M} = i g_s t^a \quad \bar{u}(k)\,\gamma\cdot\epsilon^*(q)\,u(p)$$

I will work this out for an initial left-handed quark, emitting a left-handed gluon or a right-handed gluon. (The analysis for a right-handed quark is similar.) The amplitude becomes

$$i\mathcal{M} = i g_s t^a \; u_L^\dagger(k)\,\bar{\sigma}\cdot\epsilon^*(q)\,u_L(p)$$

For our purposes, it is only necessary to work to order p_T. We need the spinors

$$u(p) = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \qquad u_k = \sqrt{2zE}\begin{pmatrix}-p_T/2zE\\1\end{pmatrix}$$

and the polarization vectors

$$\epsilon_L^*(q) = \frac{1}{\sqrt{2}}\left(0,1,i,\frac{p_T}{(1-z)E}\right) \qquad \epsilon_R^*(q) = \frac{1}{\sqrt{2}}\left(0,1,-i,\frac{p_T}{(1-z)E}\right)$$

For $q_L \to q_L g_R$, the matrix element is

$$u^\dagger\bar{\sigma}\epsilon^* u = \sqrt{2E}\sqrt{2zE}\begin{pmatrix}-\frac{p_T}{2zE} & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{p_T}{(1-z)E} & 0\\ 2 & -\frac{p_T}{(1-z)E}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= 2E\frac{\sqrt{z}}{\sqrt{2}}\left(-\frac{p_T}{(1-z)E}\right) = -\sqrt{2z}\,\frac{p_T}{(1-z)}$$

For $q_L \to q_L g_L$, the matrix element is

$$u^\dagger \bar{\sigma} \cdot \epsilon^* u = \sqrt{2E}\,\sqrt{2zE}\,\left(-\frac{p_T}{2zE}, 1\right) \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p_T}{(1-z)E} & 2 \\ 0 & -\frac{p_T}{(1-z)E} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= -\sqrt{2z}\left(-\frac{p_T}{(1-z)} - \frac{2p_T}{2z}\right) = -\sqrt{2z}\,\frac{p_T}{z(1-z)}$$

In all,

$$\sum_{s_L, s_R} |\mathcal{M}|^2 = \frac{4}{3} g_s^2 \quad 2z \frac{p_T^2}{z^2(1-z)^2} (1+z^2)$$

Now we can compute the gluon emission rate in the collinear approximation

$$\sigma(pp \to g X) = \int dx_1\, dx_2\, f_1(x_1) f_2(x_2) \frac{1}{2x_1 x_2 s} \int d\Pi\, |\mathcal{M}|^2$$

$$= \int dx_1\, dx_2\, f_1(x_1) f_2(x_2) \frac{1}{2x_1 x_2 s} \int \frac{d^3k}{(2\pi)^3 2k} \int d\Pi_X\, (2\pi)^4 \delta^{(4)}(q + k + p_2 - p_X)$$

$$\frac{4}{3} g_s^2 \quad 2z \frac{p_T^2}{z^2(1-z)^2} (1+z^2) \frac{1}{|p_T^2/(1-z)|^2} \quad |\mathcal{M}(g(k) p_2 \to X)|^2$$

In the collinear limit

$$d^3k = dz\, E\, d^2p_T = dz\, E\, dp_T^2 \cdot \pi \qquad k = (1-z)E$$

Then

$$\sigma = \int dx_1\, f(x_1) \int dx_2\, f_2(x_2) \int dz \frac{dp_T^2\, \pi}{16\pi^3 (1-z)} \frac{1}{p_T^2} \quad \frac{4}{3} g_s^2 \frac{2z}{z^2(1-z)^2} (1+z^2)(1-z)^2$$

$$\cdot \frac{z}{2 z x_1 x_2} \int d\Pi_X (2\pi)^4 \delta(k + p_2 \to X) \, |\mathcal{M}(g p_2 \to X)|^2$$

or

$$\sigma = \int dx_2 f_2(x_2) \int dx_1 f(x_1) \int dz \; \frac{dp_T^2}{p_T^2} \; \frac{4}{3}\frac{\alpha_s}{2\pi} \; \frac{1+z^2}{(1-z)} \; \sigma(q(w) p_2 \to X)$$

The momentum fraction of the quark k in the proton is

$$w = z x_1 \qquad dx_1 = \frac{dw}{z}$$

so that

$$\sigma = \int dx_2 f_2(x_2) \int_0^1 dw \int \frac{dz}{z} \; f_1\left(\frac{w}{z}\right) \; \frac{4}{3}\frac{\alpha_s}{2\pi} \int \frac{dp_T^2}{p_T^2} \; \frac{1+z^2}{(1-z)} \; \sigma(q(w) p \to X)$$

The new term that we have computed also has the form of a parton cross section. It corresponds to a modification of the quark's parton distribution

$$f_1(x) \to f_1(x) + \frac{4}{3}\frac{\alpha_s}{2\pi} \int \frac{dz}{z} \int \frac{dp_T^2}{p_T^2} \; \frac{1+z^2}{(1-z)} \; f_1\left(\frac{x}{z}\right)$$

Our formulae become inaccurate when $p_T \sim Q$, so the total change is

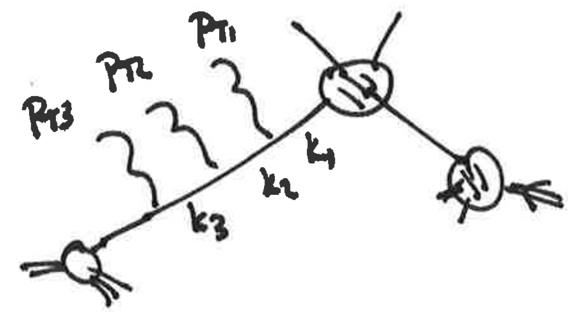
$$f_1(x) \to f_1(x) + \frac{4}{3}\frac{\alpha_s}{\pi} \log\left(Q/m_p\right) \int \frac{dz}{z} \; \frac{1+z^2}{(1-z)} \; f_1\left(\frac{x}{z}\right)$$

This is an enhancement of the quark distribution at lower values of x, proportional to the quark distribution at higher values of x. The cost of this addition is of order

$$\frac{\alpha_s(Q)}{\pi} \log Q \sim \frac{1}{2\pi} \cdot \frac{\pi}{b_0 \log Q} \cdot \log Q \sim \mathcal{O}(1)$$

That is, the emission of collinear gluons by the initial quarks is an essential aspect of the parton distribution that must be taken into account.

If we can freely emit one gluon, why not more? Consider, then, the process



Each gluon is emitted at no cost, provided that it is emitted approximately collinearly with respect to the emission at the next stage. This is the requirement

$$p_{T3} \ll p_{T2} \ll p_{T1} \ll Q$$

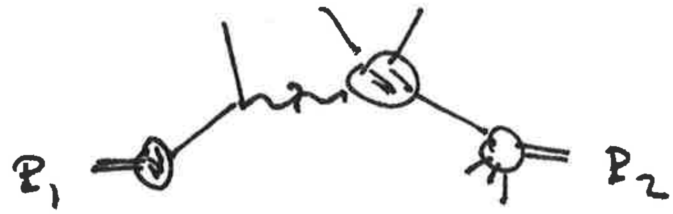
A nice way to express this is that, in every interval of p_T, there is a finite probability of gluon emission. We can consider the effect on the parton distribution as an evolution with p_T and write a differential equation to sum the effects.

Considering the process of a quark splitting off gluons only, we find the following evolution of the parton distribution:

$$\frac{d}{d\log Q} f_q(x, Q) = \frac{4}{3} \frac{\alpha_s(Q)}{\pi} \int \frac{dz}{z} \frac{1+z^2}{(1-z)} f_q\left(\frac{x}{z}, Q\right)$$

The last gluon emitted has energy close to Q, and so (with accuracy on a log scale) we can evaluate the running coupling constant α_s in this equation at the scale Q.

Quark splitting also generates a distribution of gluons that can then appear as initial-state particles in parton reactions,



The differential equation that generates this gluon distribution is

$$\frac{d}{d\log Q} f_g(x,Q) = \frac{4}{3}\frac{\alpha_s(Q)}{\pi}\int\frac{dz}{z}\,\frac{1+(1-z)^2}{z}\, f_q\!\left(\frac{x}{z},Q\right)$$

More generally, we can consider all of the collinear splitting processes

$$q \to g q \qquad \bar{q} \to g \bar{q} \qquad g \to q\bar{q} \qquad g \to g g$$

This leads to a set of coupled differential equations called the "Altarelli-Parisi" or DGLAP (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) equations. These are:

$$\frac{d}{d\log Q} f_f(x,Q) = \frac{\alpha_s(Q)}{\pi}\int\frac{dz}{z}\left[P_{q\leftarrow q}(z)\, f_f\!\left(\frac{x}{z}\right) + P_{q\leftarrow g}(z)\, f_g\!\left(\frac{x}{z}\right)\right]$$

$$\frac{d}{d\log Q} f_{\bar{f}}(x,Q) = \frac{\alpha_s(Q)}{\pi}\int\frac{dz}{z}\left[P_{q\leftarrow q}(z)\, f_{\bar{f}}\!\left(\frac{x}{z}\right) + P_{q\leftarrow g}(z)\, f_g\!\left(\frac{x}{z}\right)\right]$$

$$\frac{d}{d\log Q} f_g(x,Q) = \frac{\alpha_s(Q)}{\pi}\int\frac{dz}{z}\left[\sum_f P_{g\leftarrow q}(z)\left(f_f\!\left(\frac{x}{z}\right) + f_{\bar{f}}\!\left(\frac{x}{z}\right)\right) + P_{g\leftarrow g}(z)\, f_g\!\left(\frac{x}{z}\right)\right]$$

It is a fun and not so difficult exercise to derive the other splitting functions needed for these equations by using explicit spinors and vectors and the collinear approximation as we did above. The results for the various splitting functions are:

$$P_{g\leftarrow q} = \frac{4}{3}\,\frac{1+(1-z)^2}{z}$$

$$P_{q\leftarrow q} = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(1-z)\right]$$

$$P_{q\leftarrow g} = \frac{1}{2}\left[z^2 + (1-z)^2\right]$$

$$P_{g\leftarrow g} = 3\left[\frac{1+z^4+(1-z)^4}{z\,(1-z)_+} + \left(\frac{11}{6} - \frac{n_f}{9}\right)\delta(1-z)\right]$$

The delta functions in these formulae require some explanation. When a quark emits a gluon and moves down in x, the total number of quarks remains unchanbged. So we need to include in the splitting function a negative term proportional to $\delta(1-x)$, so that the $q \to q$ splitting function can satisfy the required conservation law

$$\int_0^1 dz \; P_{q\leftarrow q}(z) = 0$$

This is treated formally in the following way: Define a distribution

$$\frac{1}{(1-z)_+}$$

by the equation

$$\int_0^1 dz\, \frac{f(z)}{(1-z)_+} = \int_0^1 dz\, \frac{f(z)-f(1)}{(1-z)}$$

This is equivalent to

$$\frac{1}{(1-z)_+} = \lim_{\epsilon\to 0}\left\{\frac{1}{1-z}\,\Theta(1-x-\epsilon) - \delta(1-x)\int_0^{1-\epsilon} dx\, \frac{1}{1-x}\right\}$$

Then, for the formula given above, it works out that

$$\int_0^1 dz\, P_{q\to q}(z) = \frac{4}{3}\left[\int_0^1 dz\, \frac{z^2-1}{1-z} + \frac{3}{2}\right] = 0 \checkmark$$

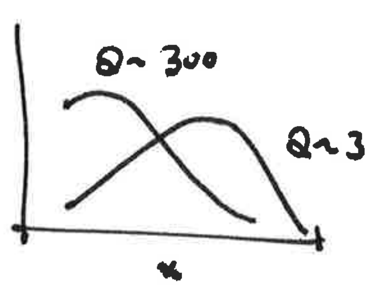
Similarly, when a gluon splits to gluons, we obtain two gluons at lower momentum while losing one gluon at higher momentum. The easiest way to account this is to insist that the gluon splitting function conserves momentum. We can check

$$\begin{aligned}
&\int_0^1 dz\, z\, P_{g\to g}(z) \\
&= \int_0^1 dz\, 3\left\{\left[\frac{1+z^4+(1-z)^4-2}{(1-z)}\right] + \frac{11}{6}\right\} \\
&= 3\int_0^1 dz\, \left\{-(1+z+z^2+z^3) + (1-z)^3 + \frac{11}{6}\right\} = 0 \checkmark
\end{aligned}$$

The final term in $P_{g\to g}(z)$ above accounts the gluons lost in gluon splitting to $q\bar{q}$. Note that this term is proportional to the number of quark flavors n_f with masses less than Q.

How do we use these equations? Imagine first that we had a concrete model of the proton structure at low Q, similar to the one described in the previous lecture. Then we would integrate the Altarelli-Parisi equations to compute the $f_i(x)$ at higher values of Q. These equations predict that the parton distribution functions vary over a logarithmic range in Q. Thus, we might not see variation in the pdfs over a small interval in Q, but we should see large effects over a large range of Q. For the LHC, we have Q values in the range 100 GeV–1 TeV, so the pdfs that we use there should be very different from those measured at small Q.

In particular, the Altarelli-Parisi equations predict three effects. First, the valence quarks distributions shift down to smaller values of x as Q increases.



The importance of the valence quarks—and the difference between protons and antiprotons as targets—goes away. Second, the emission of gluons by the valence quarks generates a gluon distribution at small x. This is by far the largest pdf at LHC energies, except at the largest values of x. Notice that the emission of gluons with small x is enhanced by a soft singularity as well as a collinear singularity, so the emission probability goes as

$$\sim \frac{4}{3}\frac{\alpha_s}{\pi}\int\frac{dz}{z}\int d\log Q$$

Third, the splitting of gluons to $q\overline{q}$ generates sea distributions for all quark flavors with masses less than Q. In particular, the heavy quarks c and b appear in addition to lighter flavors, and these can be the initial particles in parton reactions. At a 100 TeV pp collider, we will also have t and $\overline{t}$ partons in the proton beam.

We can see all of these effects in measurements of $F_2(x, Q)$ in ep and μp scattering. The required cross sections have been measured at SLAC ($Q \sim 3$ GeV), in muon scattering at CERN ($Q \sim 30$ GeV), and in ep colliding beam experiments at the HERA collider at DESY ($Q \sim 100$ GeV). The world data set for F_2 is shown in Figure 1. The striking effect of evolution on the sea distributions at very low values of x is shown in Figure 2.

This data, and other data sets from neutrino scattering and collider experiments, can be used to determine the pdfs more precisely. Without assuming an specific form a priori at an fixed value of Q, we can provide a model of pdfs with a large number of parameters, evolve this model to the Q scale of each experiment, and perform an overall fit to the data. There are several dedicated groups that now carry out sophisticated fits, including QCD corrections up to NNLO for the lepton-proton scattering processes. These include the CTEQ group, based in the US at Michigan State and SMU, the MSTW group (Martin-Stirling-Thorne-Watt) and the NNPDF group (Ball et al.), which uses a neural network as their pdf model. There is also a HERAPDF set based on clsoe analysis of the the HERA data. The results from the current NNPDF fit at two values of Q are shown in Figure 3. Comparison of these figures at low and high Q reveals all of the effects discussed above.

By convolving the pairs of pdfs, we can compute luminosity functions for pp and $p\overline{p}$ collisions. In the previous lecture, we wrote the formula for a cross section at hadron collider as

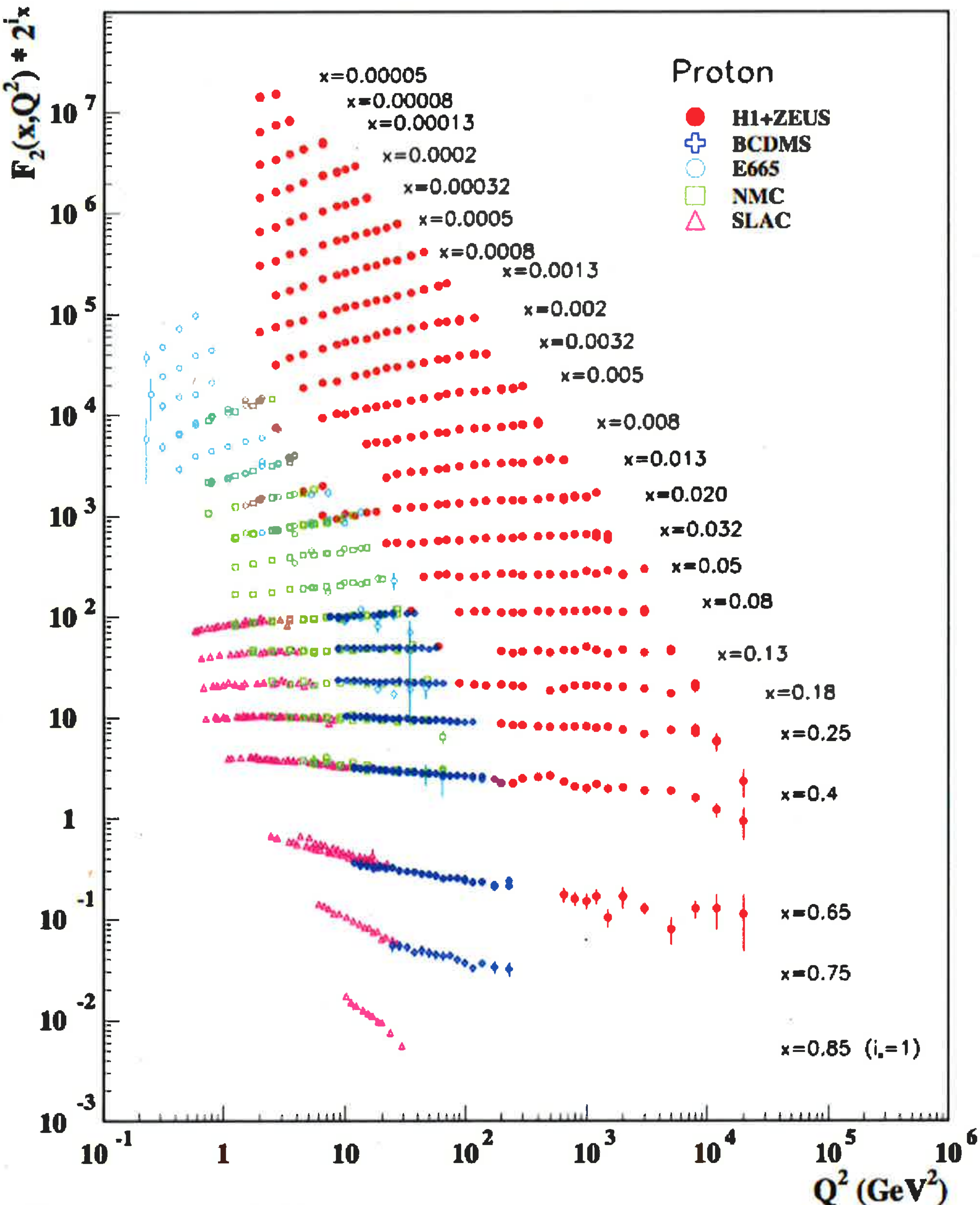


Fig. 1 Particle Data Group compilation of measurements of $F_2(x, Q^2)$, from the review of structure functions at pdg.lbl.gov

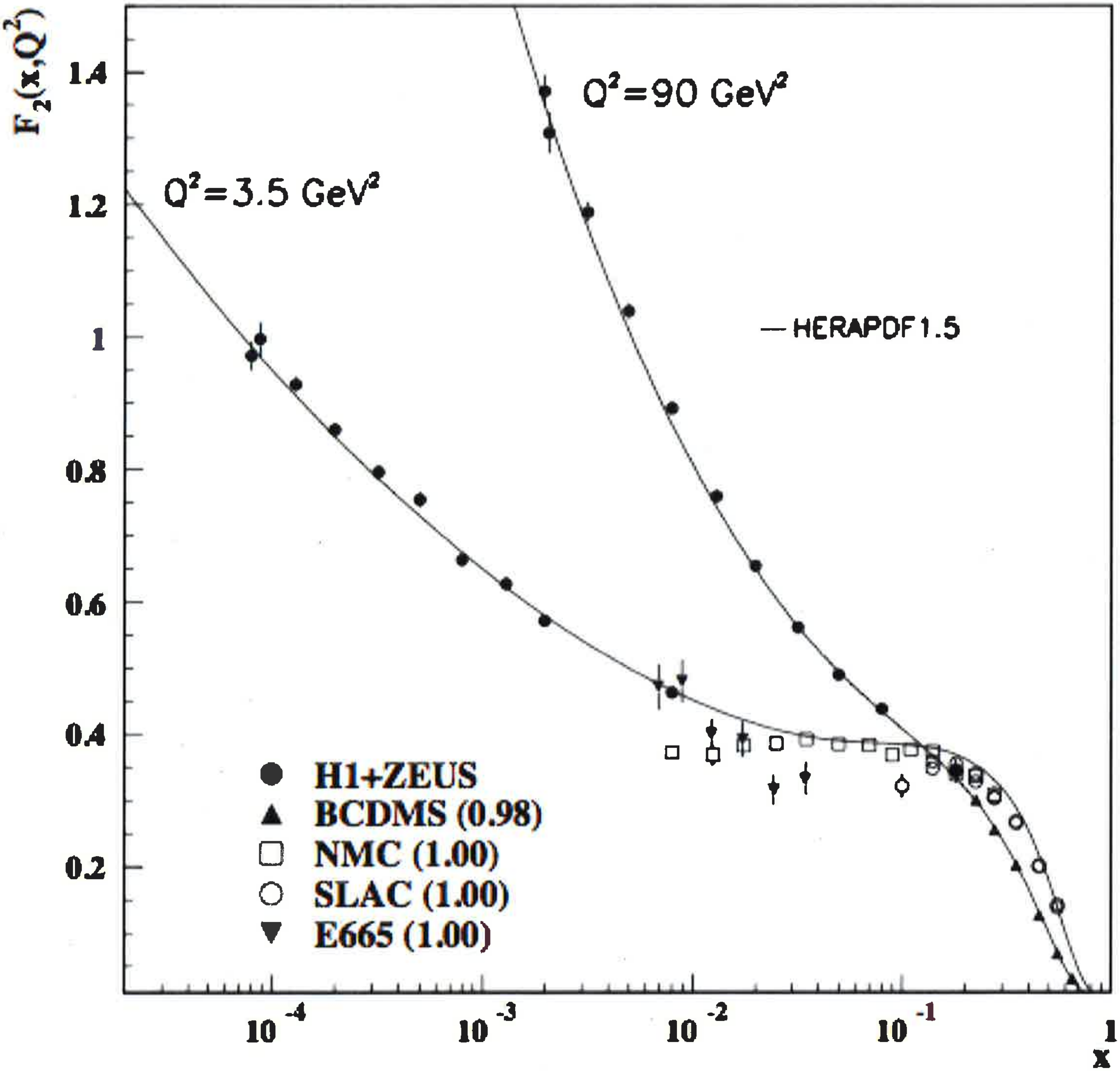


Fig. 2 Behavior of $F_2(x,Q^2)$ at very small x, as measured by the H1 and ZEUS experiments at HERA, from the Particle Data Group review of structure functions at pdg.lbl.gov

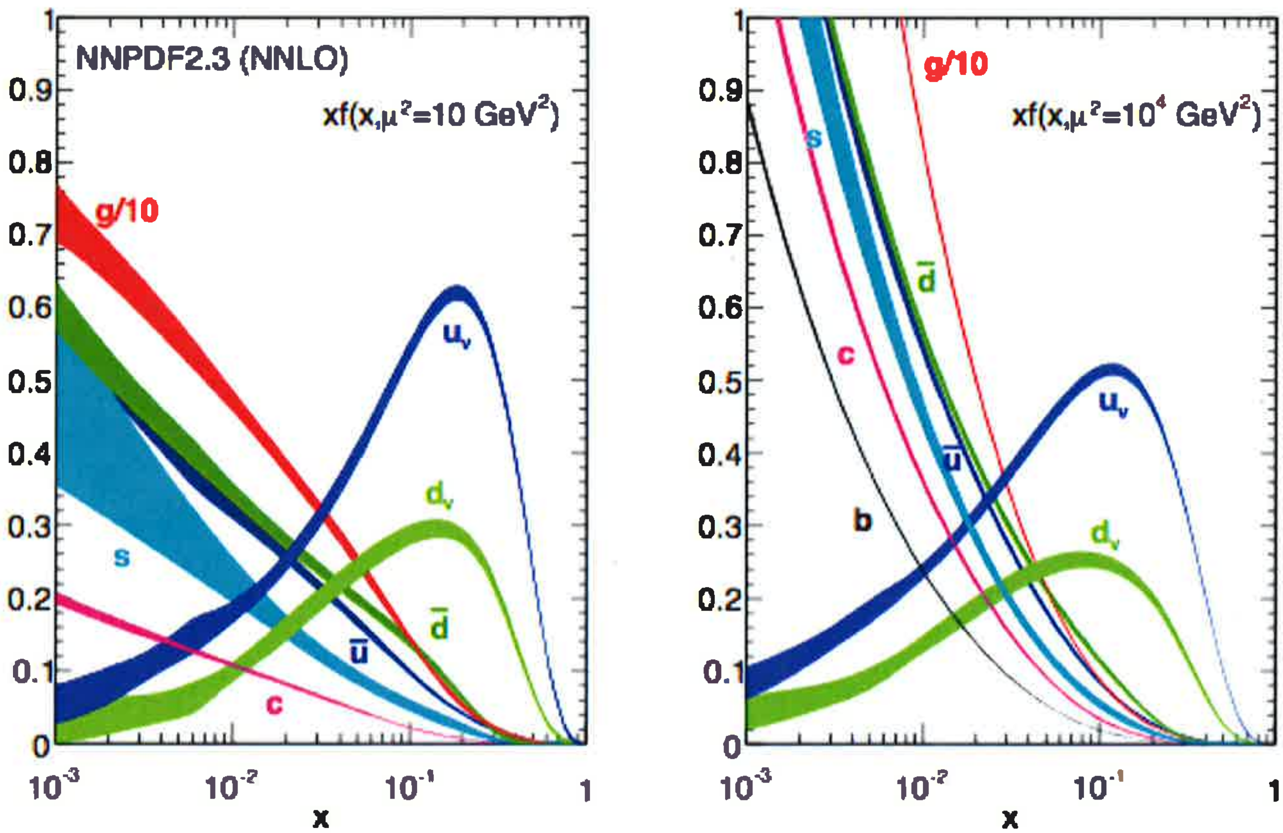


Fig. 3 Values of the pdf's at $Q = \sqrt{10}$ GeV and $Q = 100$ GeV, from the fit by the NNPDF group, from the Particle Data Group review of structure functions at pdg.lbl.gov.

$$\frac{d\sigma}{dy\, d\hat{s}}(pp \to X) = \sum_{12} \frac{x_1 f_1(x_1)\, x_2 f_2(x_2) + (f_1 \leftrightarrow f_2)}{1+\delta_{f_1 f_2}} \; \frac{1}{\hat{s}} \; \hat{\sigma}(1+2 \to X)$$

In this formula, I have taken somewhat more care with cases of identical partons, for example, uu or gg scattering. Integrating over rapidity, we have

$$\hat{s} \frac{d\sigma}{d\hat{s}} = \int dy \sum_{12} \left(\frac{x_1 f_1(x_1)\, x_2 f_2(x_2) + (f_1 \leftrightarrow f_2)}{1+\delta_{f_1 f_2}} \right) \sigma(1+2 \to X)$$

In their textbook, Ellis, Stirling, and Webber write this expression as

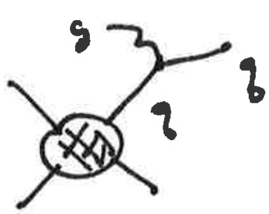
$$\hat{s} \frac{d\sigma}{d\hat{s}} = \left[\frac{1}{s} \frac{dL}{d\tau} \right] \hat{s}\, \sigma(1+2 \to X)$$

and provide graphs of the luminosity functions

$$\left[\frac{1}{s} \frac{dL}{d\tau} \right]$$

for $q\overline{q}$, gg, $q/\overline{q} + q/\overline{q}$, and $g + q/\overline{q}$ initial states. I reproduce these plots in Figures 4-7. These are log-log plots, so the behavior for $\hat{s} \ll s$ goes to a power law. This power law is approximately the rule of thumb: a factor of 10 decrease in luminosity for each factor of 2 increase in energy. (Conversely, in this regime, doubling the energy of the collider is equivalent to a factor of 10 increase in luminosity.)

The same analysis of collinear singularities applies to the final state. Consider the situation in which a quark is ejected from a hard process with momentum transfer Q. Then the quark can emit a gluon



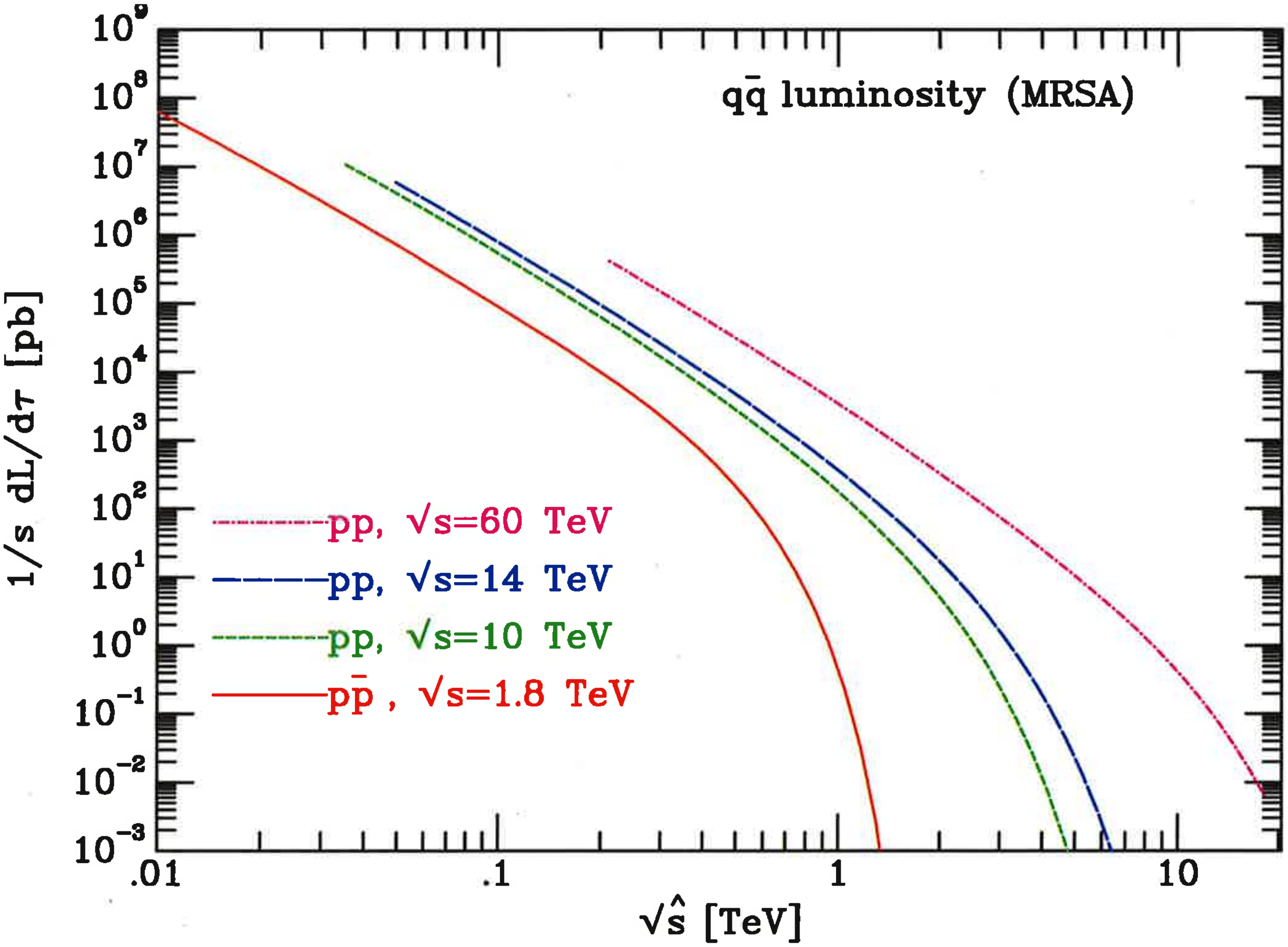


Fig. 4 Luminosity function $\left(\frac{1}{s}\frac{dL}{d\tau}\right)$ for $q\bar{q}$ processes, from Ellis, Stirling, and Webber, *QCD and Collider Physics*

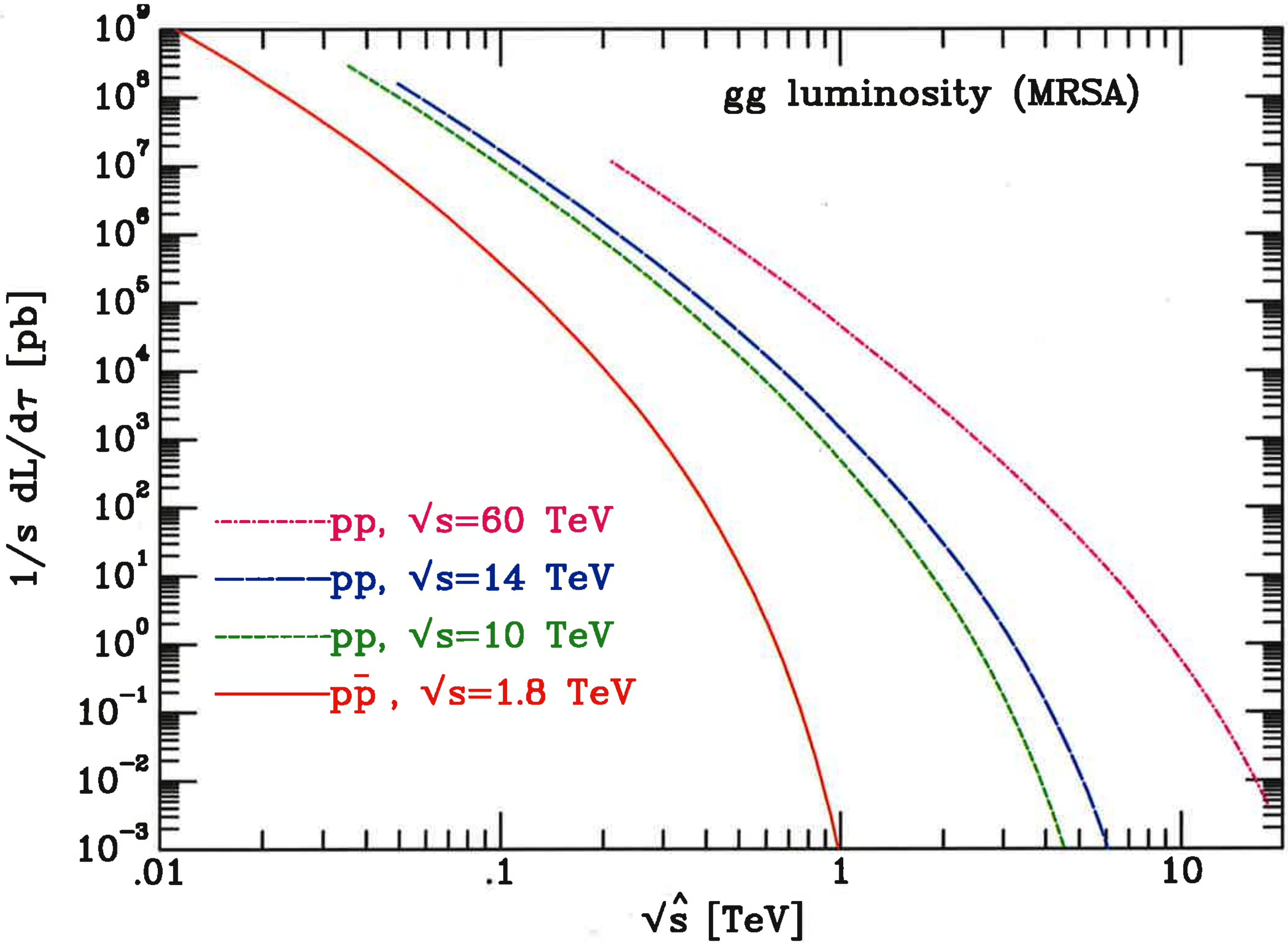


Fig. 5 Luminosity function $\frac{1}{s}\frac{dL}{d\tau}$ for gg processes,
from Ellis, Stirling, and Webber QCD and Collider Physics

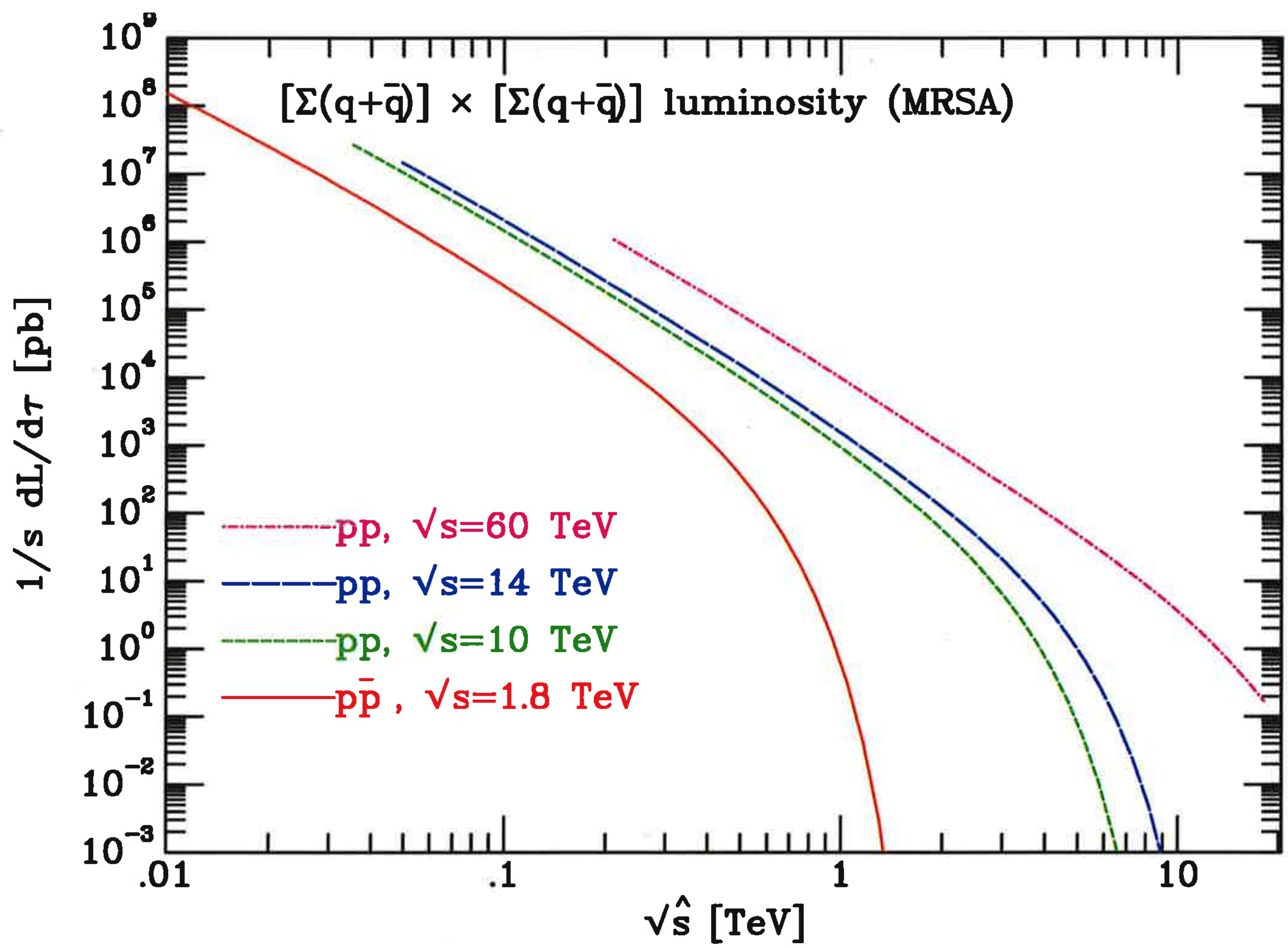


Fig. 6 Luminosity functions for $q/\bar{q}$ + $q/\bar{q}$ processes, from Ellis Stirling and Webber QCD and Collider Physics

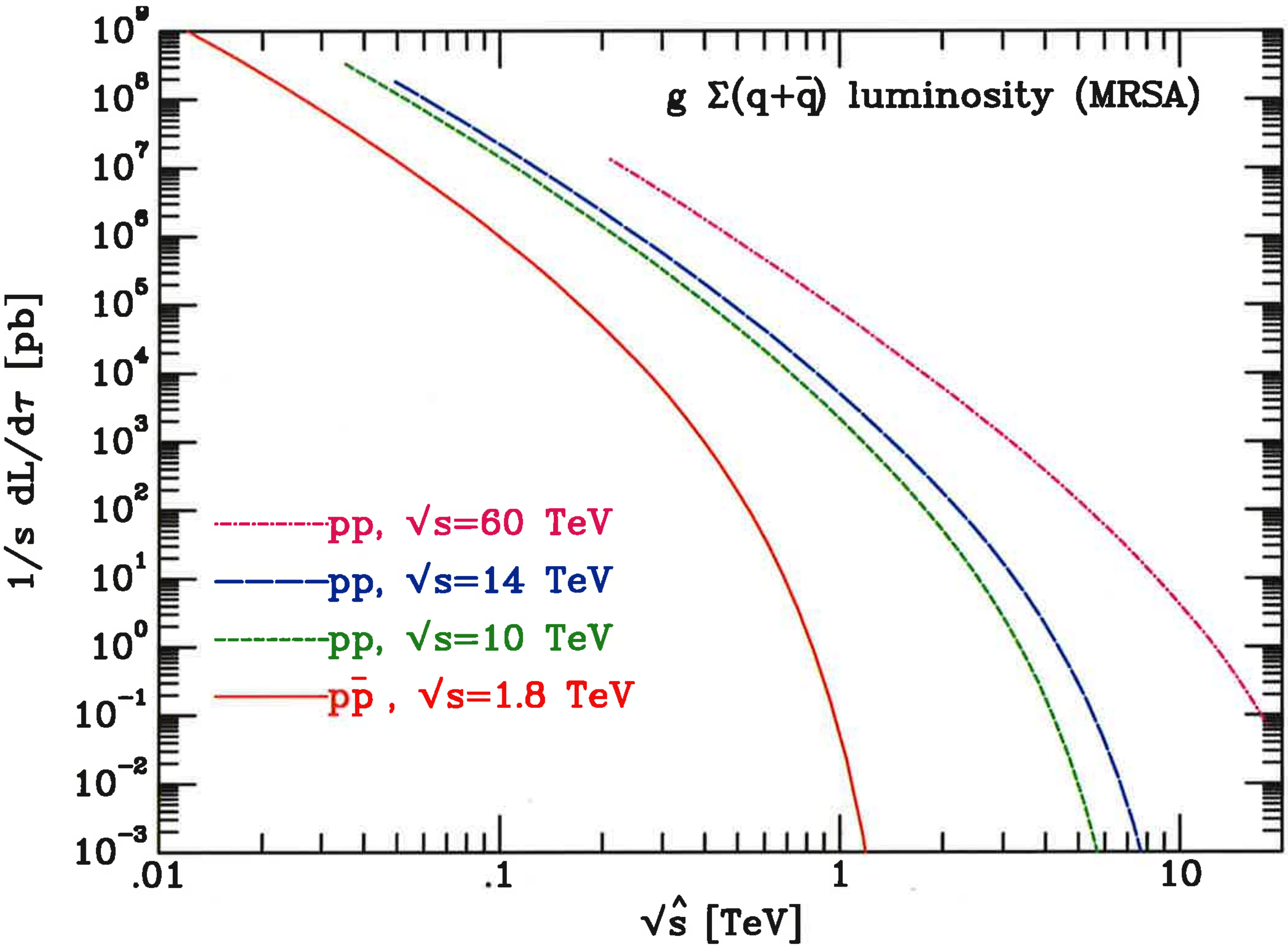


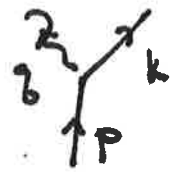
Fig. 7 Luminosity functions for $g + q/\bar{q}$ processes, from Ellis, Stirling, and Webber, QCD and Collider Physics

with a probability that is of order 1 if the gluon is approximately collinear with the original quark direction. We can work out an expression for the probability to radiate a gluon in the same way as above. The kinematics is slightly shifted, with the momentum p now off-shell

$$q = \left((1-z)E\,,\ -p_T\,,\ (1-z)E - \frac{p_T^2}{2(1-z)E}\right)$$

$$k = \left(zE\,,\ p_T\,,\ zE - \frac{p_T^2}{2zE}\right)$$

$$p = \left(E\,,\ 0\,,\ E - \frac{p_T^2}{2E\,z(1-z)}\right)$$



The value of the off-shell momentum is

$$p^2 = \frac{p_T^2}{z(1-z)}$$

The matrix element, to order p_T, is the same as that computed above. Again,

$$\sum_{s_L s_R} |\mathcal{M}|^2 = \frac{4}{3} g^2 \frac{2z\, p_T^2}{z^2(1-z)^2} (1+z^2)$$

The cross section for the process with collinear gluon emission is

$$\sigma(AB \to X g q) = \frac{1}{2s} \int d\Pi_X \frac{d^3k\, d^3q}{(2\pi)^6\, 2k\, 2q} (2\pi)^4 \delta(p_A + p_B - p_X - k - q)$$

$$\frac{4}{3} g^2 \frac{2z\, p_T^2}{z^2(1-z)^2} (1+z^2) \frac{1}{(p^2)^2} \left|\mathcal{M}(AB \to X q(p))\right|^2$$

In the collinear limit, $q = p - k$, so

$$\frac{d^3k\, d^3q}{(2\pi)^6\, 2k\, 2q} = \frac{d^3p\, d^3q}{(2\pi)^3\, 2z(1-z)E^2} = \frac{d^3p}{(2\pi)^3\, 2p} \frac{dz\, E\, dp_T^2\, \pi}{(2\pi)^3\, z(1-z)E}$$

Then

$$\sigma(AB \to X g q) = \frac{1}{2s} \int d\Pi_X \int \frac{d^3p}{(2\pi)^3 2p} (2\pi)^4 \delta(p_A + p_B - p_X - p)$$

$$|\mathcal{M}(AB \to X q)|^2 \cdot \int dz \frac{dp_T^2}{16\pi^2 z(1-z)} \frac{z^2(1-z)^2}{p_T^2} \frac{4}{3} g^2 \frac{2z p_T^2}{z^2(1-z)^2} (1+z^2)$$

so that

$$\sigma(AB \to X g q) = \sigma(AB \to X q(p)) \cdot \int dz \int \frac{dp_T^2}{p_T^2} \frac{4}{3} \frac{\alpha_s}{2\pi} \frac{1+z^2}{(1-z)}$$

The gluon emission probability is

$$\mathrm{Prob}(q \to g + q) = \frac{4}{3} \frac{\alpha_s}{\pi} \int dz \int \frac{dp_T}{p_T} \frac{1+z^2}{(1-z)}$$

This is the same structure that we found for gluon emission from the initial state. The final quark momentum distribution is given by the Altarelli-Parisi splitting function. Note that this includes a soft gluon singularity

$$\int_0 \frac{dz_g}{z_g}$$

where $z_g = (1 - z)$ is the longitudinal momentum fraction of the gluon.

It is useful to write the probability of finding the final quark at a momentum fraction z as a function of z and Q, called the "fragmentation function". As with the pdfs, the fragmentation functions evolve according to the Altarelli-Parisi equations

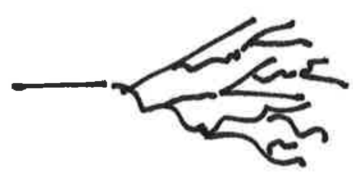
$$\frac{d}{d\log Q} f_f(z) = \frac{\alpha_s}{\pi}\int \frac{dw}{w}\left[P_{q\leftarrow q}(w)\, f_f\!\left(\frac{z}{w}\right) + P_{q\leftarrow g}(w)\, f_g\!\left(\frac{z}{w}\right)\right]$$

$$\frac{d}{d\log Q} f_{\bar f}(z) = \frac{\alpha_s}{\pi}\int \frac{dw}{w}\left[P_{q\leftarrow q}(w)\, f_{\bar f}\!\left(\frac{z}{w}\right) + P_{q\leftarrow g}(w)\, f_g\!\left(\frac{z}{w}\right)\right]$$

$$\frac{d}{d\log Q} f_g(z) = \frac{\alpha_s}{\pi}\int \frac{dw}{w}\left[P_{g\leftarrow q}(w) \sum_f \left(f_f\!\left(\frac{z}{w}\right) + f_{\bar f}\!\left(\frac{z}{w}\right)\right) + P_{g\leftarrow g}(w)\, f_g\!\left(\frac{z}{w}\right)\right]$$

The same splitting functions appear as in the initial-state equations. In particular, the delta function contributions to the splitting functions that we found above are present again, for the same reasons.

The collinear splitting process divides the momentum carried by the original quark or gluon into a sum of momenta carried by approximately collinearly moving particles.



This is a "parton shower". The transverse size of the shower is set by the first emission, whose transverse momentum scales with Q. These are the jets that we saw in event displays with quark and gluon production.

Dominantly, when we create gluons, they split to additional gluons. So quark-initiated and gluon-initiated showers will be quite similar, but with higher transverse momentum and multiplicity in the gluon case. The multiplicities should follow the probabilities for the first emission

$$\frac{n(g)}{n(q)} \sim \frac{T^2|_8}{t^2|_3} \sim \frac{8}{4/3} \sim \frac{9}{4}$$

The same structures also appear in the initial state at hadron colliders. The collinear emissions that contribute to the evolution of the pdfs also create initial-state showers along the beam direction.

In the next lecture, we will use these ideas to understand the form of typical collisions events found at the LHC.