# Lecture 3:
# Unitarity Techniques and Two-loop Integrals

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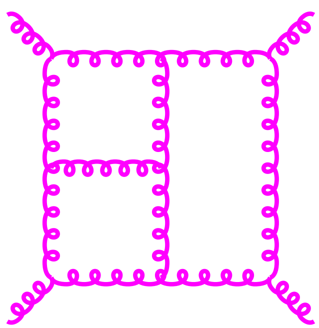
- However, standard quantizations of gauge theories and gravity, and (maximally) supersymmetric versions thereof, do not respect all the on-shell symmetries.
- Enormous intermediate expressions $\Rightarrow$ simple final answers.
- Some direct computations are beyond limit of computers.
  - One 3-loop diagram from $N = 8$ supergravity (finite??):



8 3-graviton vertices (100 terms each)

10 graviton propagators (3 terms each)

$\Rightarrow \quad 100^8 \times 3^{10} = 6 \times 10^{20}$ terms

At 2 GHz/term, $10,000$ years to complete!

# $S$ matrix unitarity

- Reconstruct quantum information — loop amplitudes from classical information — tree amplitudes by exploiting unitarity of the scattering matrix:

$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT) \qquad \Rightarrow \qquad 2\,\mathrm{Im}T = T^\dagger T$$

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- Inserting the perturbative expansion,

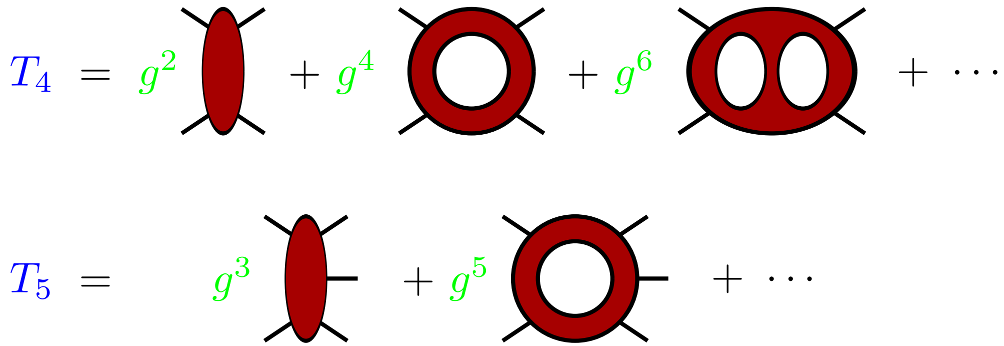
$$\begin{aligned} T_4 &= g^2\, T_4^{(0)} + g^4\, T_4^{(1)} + g^6\, T_4^{(2)} + \ldots \\ T_5 &= \qquad g^3\, T_5^{(0)} + g^5\, T_5^{(1)} + \ldots, \end{aligned}$$

yields

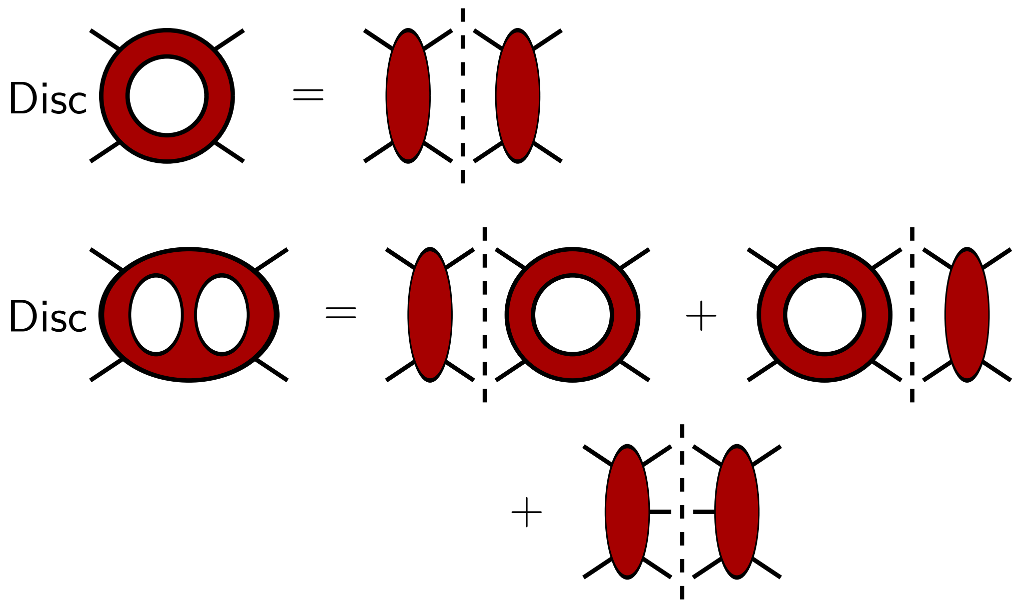
$$\begin{aligned} 2\,\mathrm{Im} T_n^{(0)} &= 0 \qquad 2\,\mathrm{Im} T_4^{(1)} = T_4^{(0)\,\dagger} T_4^{(0)} \\ 2\,\mathrm{Im} T_4^{(2)} &= T_4^{(0)\,\dagger} T_4^{(1)} + T_4^{(1)\,\dagger} T_4^{(0)} + T_5^{(0)\,\dagger} T_5^{(0)} \end{aligned}$$

# $S$ matrix unitarity graphically

Inserting




yields



# How to get $\mathrm{Re}\, T$

**Problem:** $T$ contains polynomial terms $P(s,t)$ (no cuts, no $\mathrm{Im}\, T$)

**Solutions:**

1. Dimensional regularization to the rescue:
   $D = 4 - 2\epsilon$ loop integrals $(d^{4-2\epsilon}\ell)^L$ carry fractional dimension.
   In massless gauge theory (QCD) or (super)gravity,

$$\begin{aligned} P(s,t) &\to (-s)^{-\epsilon L} P(s,t) \\ &= P(s,t) - \epsilon L \ln(-s) P(s,t) + \ldots \end{aligned}$$

   Polynomials acquire imaginary parts, one higher order in $\epsilon$.

2. Use other analytic properties, especially collinear limits
3. Invoke supersymmetry

# Building blocks: color-ordered amplitudes

First color decompose tree amplitudes.

For simplicity, let all particles be in adjoint representation.

Color factors at tree-level are all of form $f^{a_{i_1} a_{i_2} b} f^{b a_{i_3} a_{i_4}}$. Using

$$f^{abc} = \frac{-i}{\sqrt{2}} \Big( \mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^a T^c T^b) \Big)$$

plus color Fierz identity

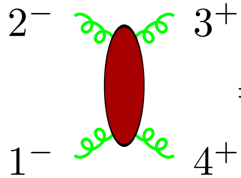
$$(T^a)_{i_1}^{\;\bar{\jmath}_1} \, (T^a)_{i_2}^{\;\bar{\jmath}_2} \; = \; \delta_{i_1}^{\;\bar{\jmath}_2} \delta_{i_2}^{\;\bar{\jmath}_1} - \frac{1}{N_c} \, \delta_{i_1}^{\;\bar{\jmath}_1} \delta_{i_2}^{\;\bar{\jmath}_2}$$

can express any tree amplitude as

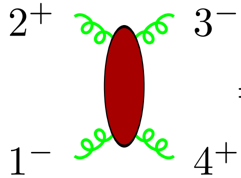
$$\mathcal{A}_n^{\mathrm{tree}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n / Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) \, A_n^{\mathrm{tree}}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n}))$$

# Building blocks — in $D = 4$

Color-ordered (“primitive”) tree-level helicity amplitudes $A_4^{\text{tree}}$:




$$= i \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle} = \text{phase} \times \frac{s_{12}}{s_{23}}$$

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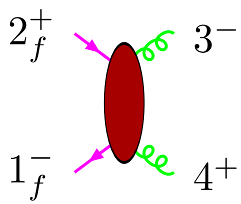


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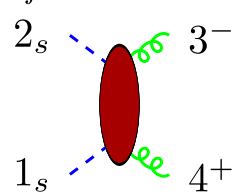
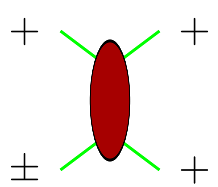



$$= i \frac{\langle 1\,3\rangle^2 \langle 2\,3\rangle^2}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle}$$




$= 0$ (any states)

Spinor products $\langle i\, j\rangle \equiv \bar{u}_L(k_i) u_R(k_j) = \text{phase} \times \sqrt{s_{ij}}$

Tree amplitude simplicity due to supersymmetry

# Supersymmetry Ward identities

Grisaru, Pendleton, van Nieuwenhuizen (1977)

- In any unbroken supersymmetric theory, the supercharge $Q$ annihilates the vacuum, so

$$0 = \langle 0|[Q, \Phi_1 \Phi_2 \cdots \Phi_n]|0\rangle = \sum_{i=1}^{n} \langle 0|\Phi_1 \cdots [Q, \Phi_i] \cdots \Phi_n|0\rangle$$

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- Let $\Phi_i$ make helicity eigenstates $\Rightarrow$ many terms vanish:

$$A_n^{\mathrm{SUSY}}(1^\pm, 2^+, 3^+, 4^+, \ldots, n^+) = 0$$

$$A_n^{\mathrm{SUSY}}(1_P^-, 2_P^+, 3^-, 4^+, \ldots, n^+)$$

$$= \left( \frac{\langle 1\, 3 \rangle}{\langle 2\, 3 \rangle} \right)^{2|h_P|} A_n^{\mathrm{SUSY}}(1_{\bar\phi}, 2_\phi, 3^-, 4^+, \ldots, n^+)$$

$\phi =$ scalar; $\quad h_P =$ helicity of particle $P$

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  — must pair produce them
- hence fermions could have been
  in adjoint representation (gluinos)
- at loop-level, QCD "discovers" that it is
  not supersymmetric
- still, special linear combinations of color-decomposed
  QCD amplitudes are supersymmetric

# Example: 1-loop QCD $\beta$ function

- $\beta < 0 \Rightarrow$ asymptotic freedom

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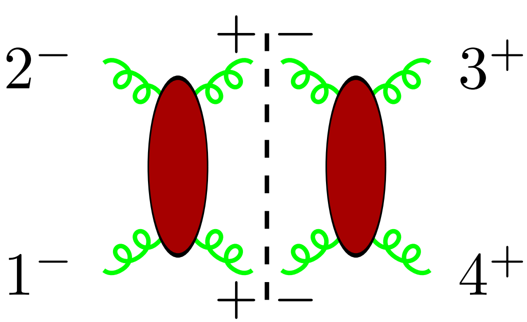
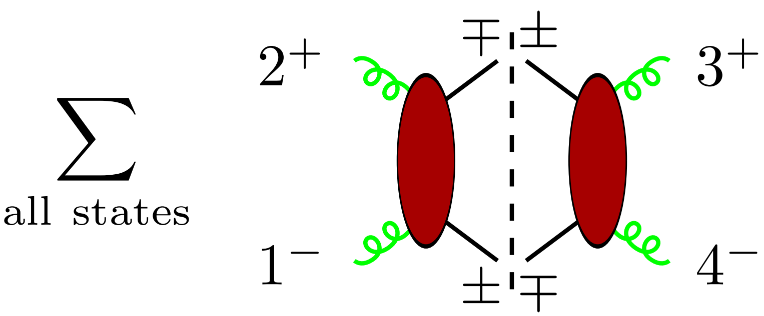
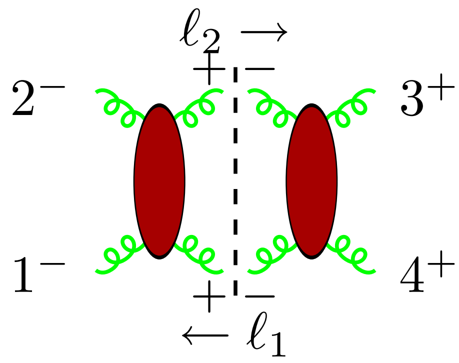
- 1-loop ultraviolet divergence $\Leftrightarrow 1/\epsilon$ pole;
  can use $D = 4$ tree helicity amplitudes in cuts
- Two cuts to evaluate:



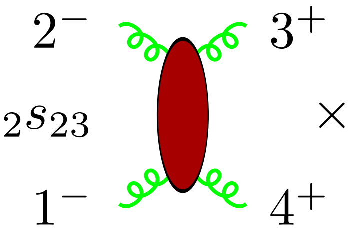



# First cut

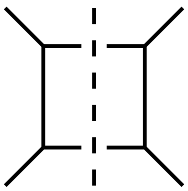



$$= \text{phase} \times \frac{s_{12}}{(\ell_1 - k_1)^2} \times \frac{s_{12}}{(\ell_2 - k_3)^2}$$

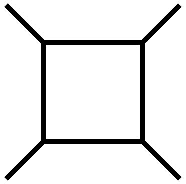
$$= i\, s_{12} s_{23}$$




$$\times$$



- All loop momenta in numerators come out of integral!
- Only a (cut) scalar box integral, as in $\phi^3$ theory:



$$= \int \frac{d^{4-2\epsilon}\ell_1}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell_1^2(\ell_1 - k_1)^2(\ell_1 - k_1 - k_2)^2(\ell_1 + k_4)^2}$$

- No ultraviolet divergence; no contribution to $\beta$ function.

# Interlude: Spinor identities

- Definitions: $\langle i\, j\rangle \equiv \bar{u}_L(k_i)u_R(k_j)$, $[i\, j] \equiv \bar{u}_R(k_i)u_L(k_j)$; $k_i^2 = k_j^2 = 0$

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- Normalization: $\langle i\, j\rangle\, [j\, i] = \bar{u}_L(k_i)u_R(k_j)\bar{u}_R(k_j)u_L(k_i)$
  $= \mathrm{Tr}[\frac{1}{2}(1-\gamma_5)\,\not{k}_i\,\not{k}_j] = 2k_i\cdot k_j \equiv s_{ij} = \langle i\, j\rangle\,[j\, i]$

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- Momentum conservation: If $\sum_{i=1}^{n} k_i^\mu = 0$, then $\bar{u}_R(k_j)\sum_{i=1}^{n}\not{k}_i u_R(k_m) = \sum_{i=1}^{n}[j\, i]\,\langle i\, m\rangle = 0$

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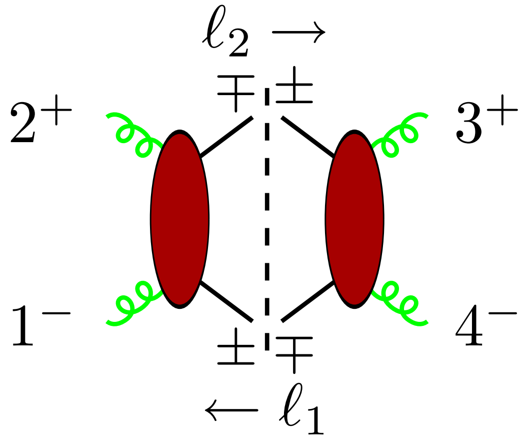
  $\bar{u}_R(k_j)\sum_{i=1}^n \not{k}_i u_R(k_m) = \sum_{i=1}^n [j\, i]\langle i\, m\rangle = 0$
- Schoutens: $\langle i\, j\rangle \langle \ell\, m\rangle [m\, \ell] = \langle i\, j\rangle s_{\ell m}$
  $= u_L(k_i)(\not{k}_\ell \not{k}_m + \not{k}_m \not{k}_\ell)u_R(k_j)$
  $= \langle i\, \ell\rangle [\ell\, m]\langle m\, j\rangle + \langle i\, m\rangle [m\, \ell]\langle \ell\, j\rangle$
  $\Rightarrow \quad \langle i\, j\rangle \langle \ell\, m\rangle = \langle i\, \ell\rangle \langle j\, m\rangle + \langle i\, m\rangle \langle \ell\, j\rangle$

# Second cut

$$\sum_{\text{all states}}$$




$$= \frac{i^4}{\langle 1\,2\rangle\,\langle 2\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle\,\langle \ell_1\,1\rangle\;\langle \ell_2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,\ell_1\rangle\,\langle \ell_1\,\ell_2\rangle}$$

$$\times \Bigg[ \langle \ell_1\,1\rangle^4 \langle \ell_2\,4\rangle^4 + \langle \ell_2\,1\rangle^4 \langle \ell_1\,4\rangle^4$$

$$+ \frac{N_f}{N_c}\Big( -\langle \ell_1\,1\rangle^3\,\langle \ell_2\,1\rangle\,\langle \ell_2\,4\rangle^3\,\langle \ell_1\,4\rangle$$

$$- \langle \ell_1\,1\rangle\,\langle \ell_2\,1\rangle^3\,\langle \ell_2\,4\rangle\,\langle \ell_1\,4\rangle^3 \Big)$$

$$+ \frac{N_s}{N_c}\langle \ell_1\,1\rangle^2 \langle \ell_2\,1\rangle^2 \langle \ell_2\,4\rangle^2 \langle \ell_1\,4\rangle^2 \Bigg]$$

| | helicity |
|---|---|
| gluons | $\pm 1$ |
| quarks | $\pm\frac{1}{2}$ |
| scalars | $0$ |

# Second cut (cont.)

$$= \frac{i^4}{\langle 1\,2\rangle\,\langle 2\,\ell_2\rangle\,\langle \ell_2\,\ell_1\rangle\,\langle \ell_1\,1\rangle\;\;\langle \ell_2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,\ell_1\rangle\,\langle \ell_1\,\ell_2\rangle}$$

$$\times \Big[\langle \ell_1\,\ell_2\rangle^4 \langle 1\,4\rangle^4 \qquad N=4$$

$$+ \left(\frac{N_f}{N_c} - 4\right)\left(-\langle \ell_1\,1\rangle\,\langle \ell_2\,1\rangle\,\langle \ell_2\,4\rangle\,\langle \ell_1\,4\rangle\;\;\langle \ell_1\,\ell_2\rangle^2 \langle 1\,4\rangle^2\right) \qquad N=1$$

$$+ \left(\frac{N_s}{N_c} - 2\frac{N_f}{N_c} + 2\right)\langle \ell_1\,1\rangle^2 \langle \ell_2\,1\rangle^2 \langle \ell_2\,4\rangle^2 \langle \ell_1\,4\rangle^2\Big] \qquad \text{scalar}$$

rearranged using Schouten identity,

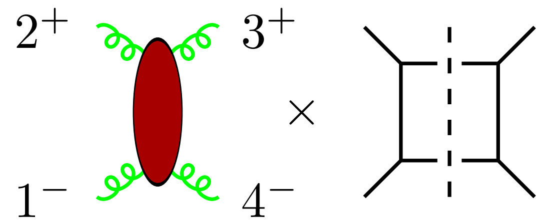
$$\langle \ell_1\,1\rangle\,\langle \ell_2\,4\rangle - \langle \ell_2\,1\rangle\,\langle \ell_1\,4\rangle = \langle \ell_1\,\ell_2\rangle\,\langle 1\,4\rangle$$

Note that $(\ell_1 - \ell_2)^2 = (k_1 + k_2)^2$, so $|\,\langle \ell_1\,\ell_2\rangle\,| = |\,\langle 1\,2\rangle\,|$.

# Second cut (cont.)

$N = 4$ term is just

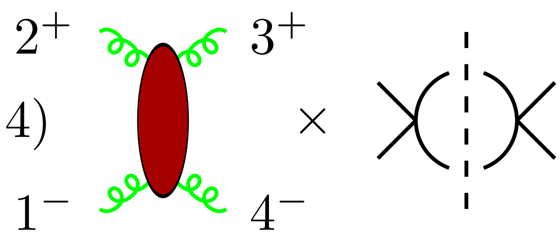
$i\, s_{12} s_{23}$




(again drop for $\beta$ function)

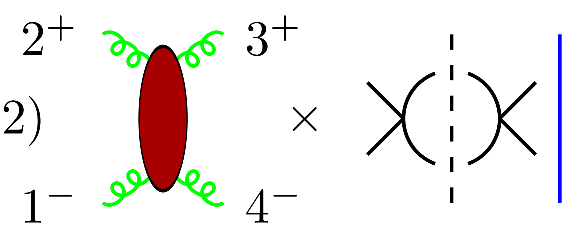
On-shell algebra for $N = 1$ term $\Rightarrow$

$-i\,(\frac{N_f}{N_c} - 4)$




and scalar term $\Rightarrow$

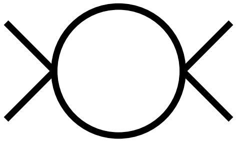
$-\frac{i}{s_{12}}\,(\frac{N_s}{N_c} - 2\frac{N_f}{N_c} + 2)$




$\Big|_{\epsilon \to \epsilon - 1}$

# QCD $\beta$ function (cont.)

Only one loop integral required, the two-point integral



$$= \int \frac{d^{4-2\epsilon}\ell_1}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell_1^2(\ell_1 - k_1 - k_2)^2}$$

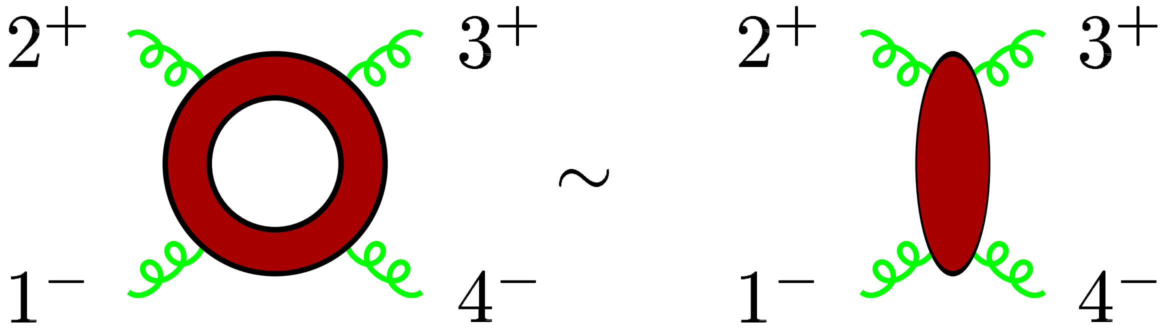
$$= \frac{i}{(4\pi)^{2-\epsilon}} \times \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} (-s_{12})^{-\epsilon}$$

$$\to \frac{i}{(4\pi)^{2-\epsilon}} \times \frac{1}{\epsilon} \qquad N=1 \text{ case}$$

$$\to \frac{i}{(4\pi)^{2-\epsilon}} \times \frac{s_{12}}{6\epsilon} \qquad \text{scalar case } (\epsilon \to \epsilon - 1)$$

# QCD $\beta$ function (cont.)

Thus



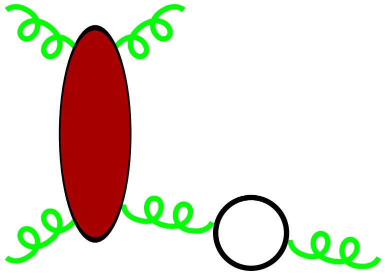

$$\times \frac{i}{(4\pi)^{2-\epsilon}} \frac{N_c}{\epsilon} \left[ \left( \frac{N_f}{N_c} - 4 \right) + \frac{1}{6} \left( \frac{N_s}{N_c} - 2\frac{N_f}{N_c} + 2 \right) \right]$$

or

$$\times \frac{i}{(4\pi)^{2-\epsilon}} \frac{1}{\epsilon} \left[ -\frac{11}{3} N_c + \frac{2}{3} N_f + \frac{1}{6} N_s \right]$$

# Right form, wrong sign!

- $S$-matrix elements have infrared divergences too, which $D = 4 - 2\epsilon$ also regulated.
- We dropped some of them, the “soft” ones, $\sim 1/\epsilon^2$, in the scalar box integral.
- But there are massless external bubbles, 0 in dim. reg., in which IR & UV collinear $\epsilon$ poles cancel exactly:



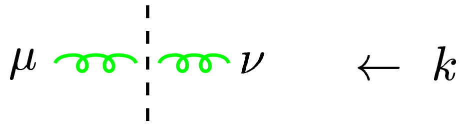
- These can be computed easily using collinear limits of tree amplitudes. Accounting for them flips the sign of the $\epsilon$ pole to the correct one.

# Beyond the $\beta$ function in QCD

- We actually computed more than the 1-loop QCD $\beta$ function.
- At 1-loop, the $D = 4$ cuts suffice to obtain the full amplitude (through $\mathcal{O}(\epsilon^0)$) in supersymmetric theories ($N = 4$ or $N = 1$). *I.e.*, no polynomial ambiguity. Bern, LD, Dunbar, Kosower
- So we have computed the $N = 4$ or $N = 1$ pieces of the leading-color terms in $A_4^{\text{1-loop}}(1_g^-, 2_g^+, 3_g^+, 4_g^-)$ — one of the two helicity configurations needed for $gg \to gg$ at NLO accuracy; the other is $A_4^{\text{1-loop}}(1_g^-, 2_g^+, 3_g^-, 4_g^+)$.
- Subleading-color terms are given by permutations of leading-color terms (at 1-loop). Bern, LD, Dunbar, Kosower
- What about the scalar piece?
- For this one needs to evaluate cuts in $D = 4 - 2\epsilon$.

# Cuts in $D = 4 - 2\epsilon$

- For $D = 4$ momenta, label physical gluon states by helicity:



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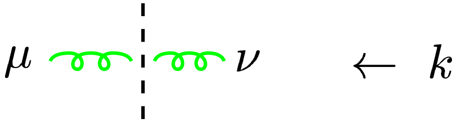


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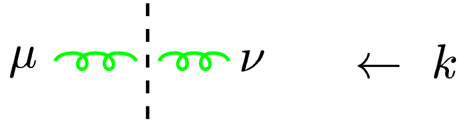


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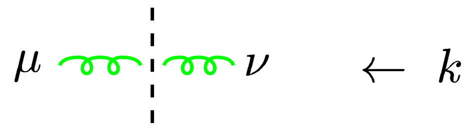


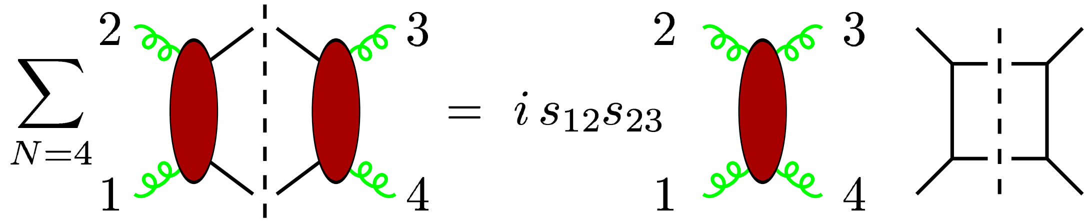

- Physical state projector is $P^{\mu\nu}_{D=4}(k) = \sum_{\lambda=\pm} \varepsilon^{\mu}_{-\lambda}(-k)\varepsilon^{\nu}_{\lambda}(k)$
- Can use simple properties of $D = 4$ tree amplitudes.
- For momenta outside $D = 4$, *e.g.* for $\mathcal{O}(\epsilon)$ cuts, helicity sum is insufficient. Use more general axial gauge projector

$$P^{\mu\nu}_{D}(k, q) = -\eta^{\mu\nu} + \frac{k^{\mu}q^{\nu} + q^{\mu}k^{\nu}}{k \cdot q}$$

(*c.f.* interference method). Results independent of light-like vector $q^{\mu}$, but algebra not as simple as in $D = 4$.

# Cuts for $N = 4$ super-Yang-Mills theory

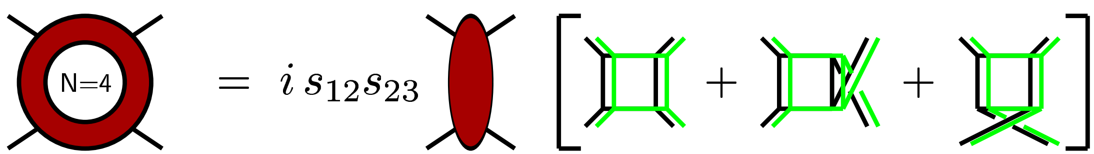
- From $D = 4$ cuts for QCD $\beta$ function, observe that




- Actually hold for $D = 4 - 2\epsilon$ and for any set of $N = 4$ external states.

Green, Schwarz, Brink

- Full amplitude, dressed with color:

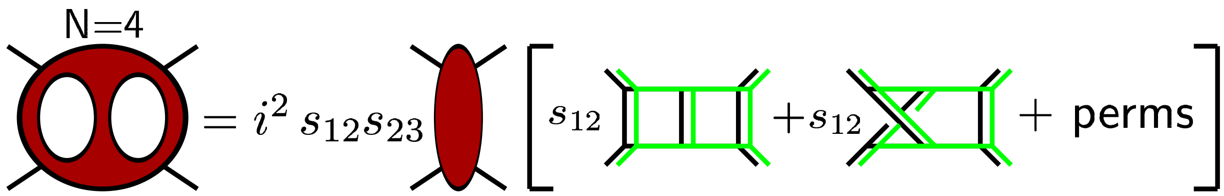



where


- Note that amplitude is proportional to tree amplitude.

# $N = 4$ SYM at two loops

- For two-loop amplitude, recycle 2-particle cuts, confirm with 3-particle cuts: Bern, Rozowsky, Yan




- Again all numerator loop momenta can be extracted!
- Scalar double box integrals UV behavior:

$$\sim \int \frac{d^D\ell_1 \, d^D\ell_2}{(\ell_i^2)^7} \sim \ell_i^{2(D-7)}$$

$\Rightarrow$ amplitude manifestly ultraviolet finite until $D = 7$.

$D = 6, 7$ divergences agrees with previous Feynman diagram result, cancellations more manifest. Marcus, Sagnotti (1985)

# $N = 8$ supergravity at two loops

- Particle content of $N = 8$ supergravity $=$ "square" of $N = 4$ super=Yang-Mills theory. (256 = 16 $\times$ 16)

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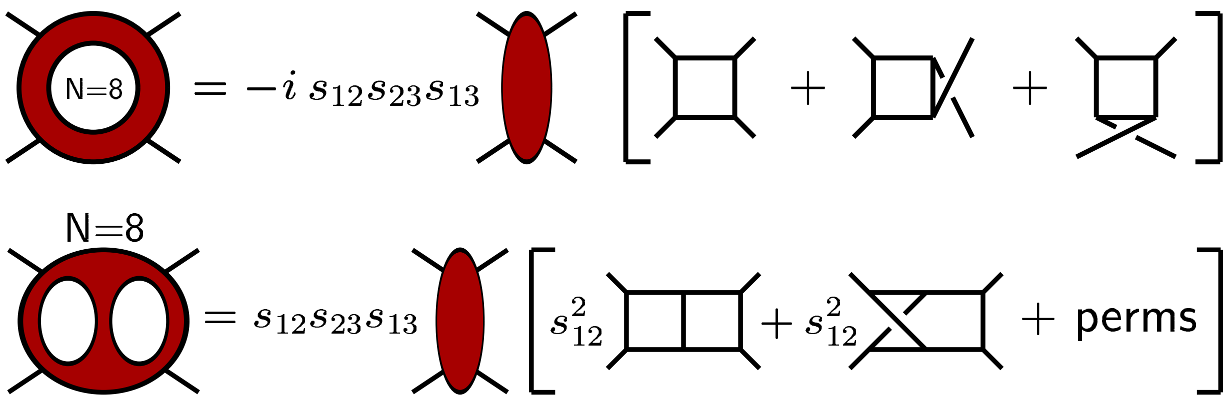

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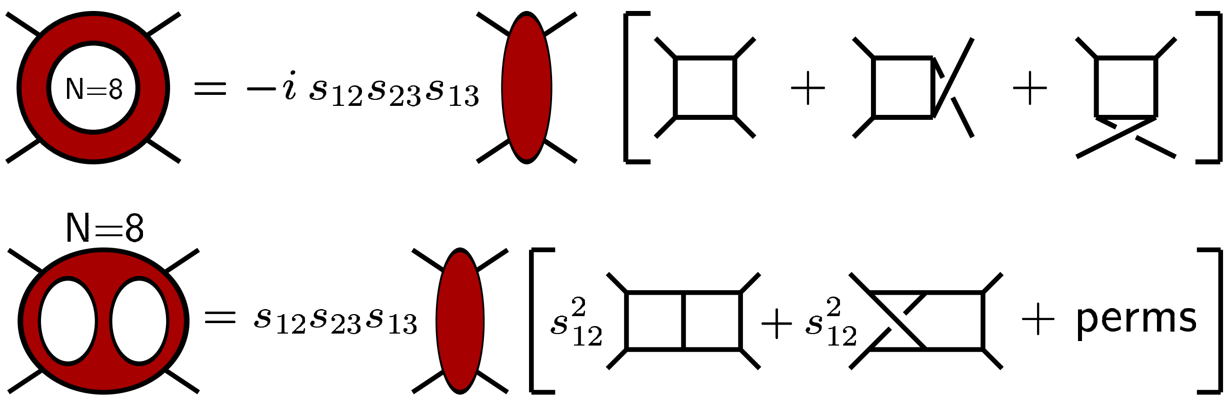
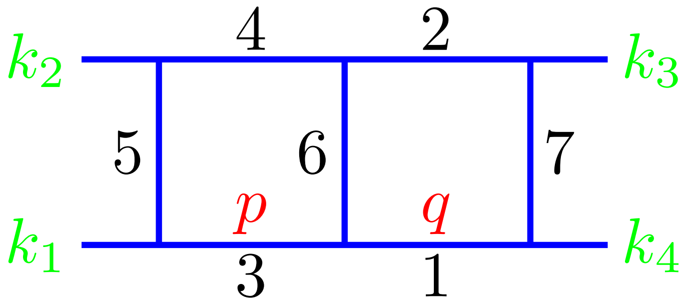



- UV behavior "better than anticipated".

Bern et al.

# Two-loop integrals

- To go any further, we need to evaluate 2-loop 4-point integrals as a Laurent series in $\epsilon$.
- Except for special cases, like $N = 4$ super-Yang-Mills and $g^+ g^+ g^+ g^+$, Bern, LD, Kosower get many tensors in the loop momentum in the numerator. *E.g.* for planar double box integral,




$$\mathcal{I}[P] \equiv \int d^D p \, d^D q \, \frac{P(p^\mu, q^\nu; k_i)}{p_1^2 \, p_2^2 \cdots p_7^2}$$

# Schwinger parametrization

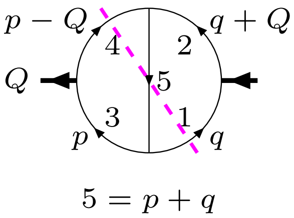
- Formula $$\frac{1}{(p_i^2)^{\nu_i}} = \frac{1}{\Gamma(\nu_i)} \int_0^\infty dt_i\, t_i^{\nu_i - 1} \exp(-t_i p_i^2)\,,$$
  First use for all $\nu_i = 1$.
  Then shift $p^\mu, q^\mu$ to diagonalize quadratic form in Gaussian integral. Find that

$$p^\mu,\ q^\mu \ \rightarrow\ (\text{quadratic in } t_i) k_j^\mu$$

- Now use formula in reverse for $\nu_i > 1$.
- $\Rightarrow$ Can trade tensor integrals for scalar integrals with multiple propagators,

$$\mathcal{I}(\nu_1, \ldots, \nu_7) \equiv \int \frac{d^D p\, d^D q}{(p_1^2)^{\nu_1} (p_2^2)^{\nu_2} \cdots (p_7^2)^{\nu_7}} \qquad \nu_i = 0, 1, 2, \ldots$$

# Integration by parts Tkachov (1981), Chetyrkin & Tkachov (1981)




A

A simple two-loop example:
Consider

$$
\begin{aligned}
0 &= \int d^D p d^D q \frac{\partial}{\partial p^\mu} \left[ \frac{p^\mu + q^\mu}{(q^2)^{\nu_1} ((q+Q)^2)^{\nu_2} (p^2)^{\nu_3} ((p-Q)^2)^{\nu_4} ((p+q)^2)^{\nu_5}} \right] \\
&= \int d^D p d^D q \frac{1}{(q^2)^{\nu_1} ((q+Q)^2)^{\nu_2} (p^2)^{\nu_3} ((p-Q)^2)^{\nu_4} ((p+q)^2)^{\nu_5}} \\
&\quad \times \left[ \nu_3 \frac{q^2 - (p+q)^2}{p^2} + \nu_4 \frac{(q+Q)^2 - (p+q)^2}{(p-Q)^2} + (D - \nu_3 - \nu_4 - 2\nu_5) \right]
\end{aligned}
$$

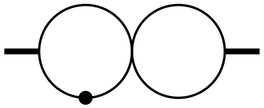
or

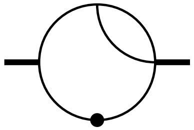
$$
0 = -\nu_3 \mathbf{3}^+ \mathbf{5}^- + \nu_3 \mathbf{1}^- \mathbf{3}^+ - \nu_4 \mathbf{4}^+ \mathbf{5}^- + \nu_4 \mathbf{2}^- \mathbf{4}^+ + (D - \nu_3 - \nu_4 - 2\nu_5)
$$

# IBP example (cont.)

- To solve for scalar integral, set all $\nu_i = 1 \Rightarrow$

$$I(1,1,1,1,1) = \frac{1}{2\epsilon}\Big[-I(1,1,2,1,0) + I(0,1,2,1,1) - I(1,1,1,2,0) + I(1,0,1,2,1)\Big]$$

$$I(1,1,2,1,0) =$$ 

$$I(0,1,2,1,1) =$$ 

- Right-hand-side has simpler topologies — propagators cancelled to give “boundary integrals”.

# Back to the double box

- Consider the $2 \times 5 = 10$ equations

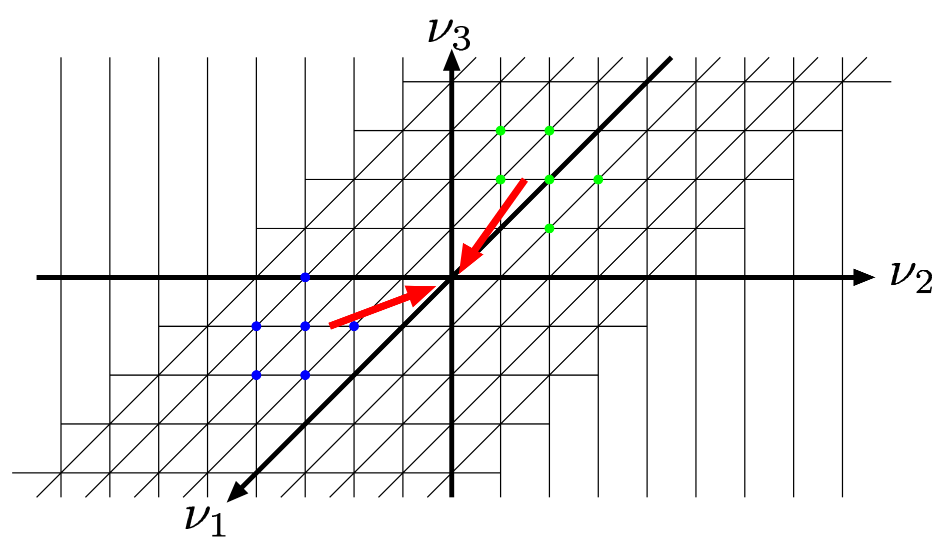
$$0 = \int d^D p \, d^D q \, \frac{\partial}{\partial \ell^\mu} \frac{b^\mu}{(p_1^2)^{\nu_1}(p_2^2)^{\nu_2} \cdots (p_7^2)^{\nu_7}} \, ; \quad \ell^\mu = p^\mu, q^\mu, \quad b^\mu = p^\mu, q^\mu, k_i^\mu$$

— also 3 from Lorentz invariance

Gehrmann, Remiddi

$\Rightarrow$ 13 linear equations for each point

$$0 = \sum_{\{\nu_i\}} c_M(\nu_1, \ldots, \nu_7) \mathcal{I}(\nu_1, \ldots, \nu_7), \quad M = 1, 2, \ldots, 13$$

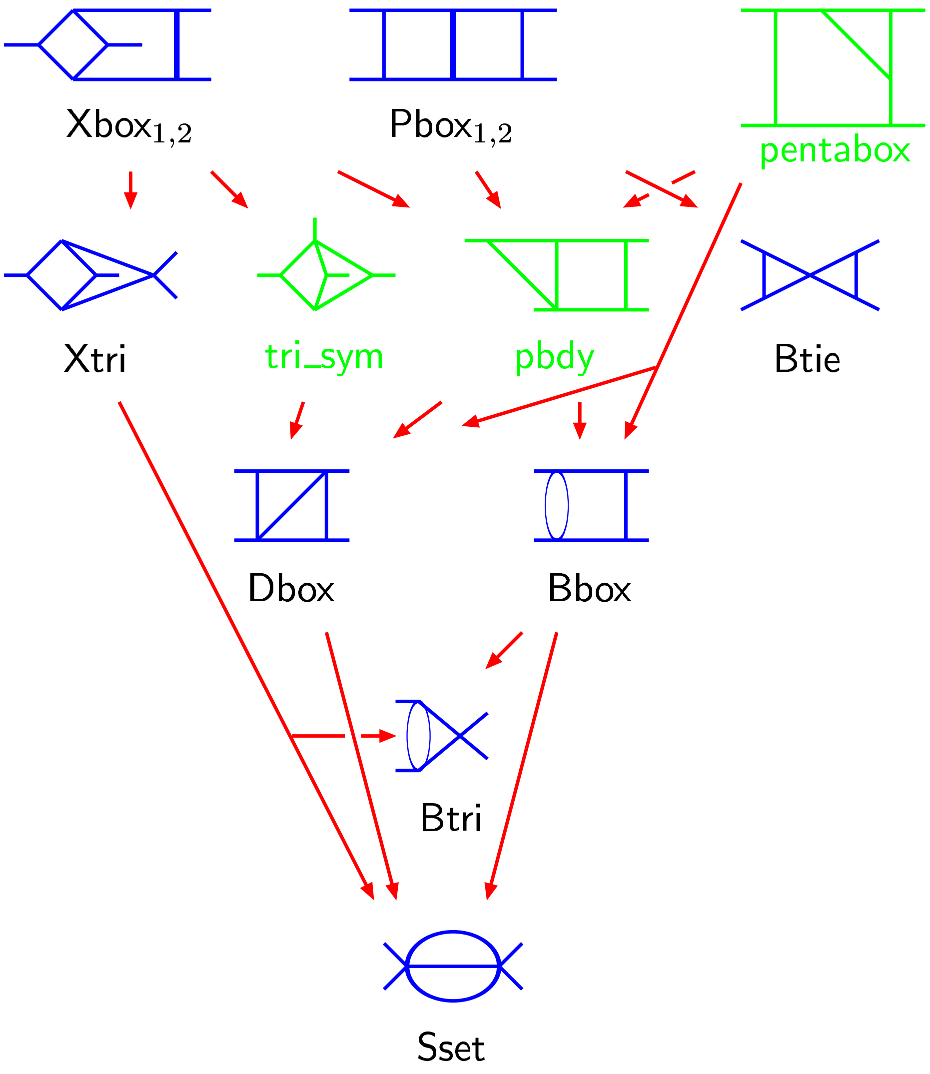


# IBP (cont.)

- Want to solve equations in direction of origin, in terms of irreducible master integrals.
- In some cases, algorithmic solutions are available, valid for any $\{\nu_i\}$.
  - All-massless planar double box — Smirnov, Veretin
  - All-massles nonplanar double box — Anastasiou et al.
- Another approach uses the fact that number of equations grows faster than number of unknowns $\mathcal{I}(\nu_i)$, as one works outward from the origin $\{\nu_i\} = \mathbf{0}$.

  Laporta; Laporta & Remiddi; Gehrmann & Remiddi
- Solve equations only as far out as you need them. (See next lecture.)

# All-massless planar subtopologies



# Master integrals

- Several techniques available:
  - Direct integration
  - Mellin-Barnes representation ($\to$ contour integrals nested sums)
  - Differential equations, using IBP system (see next lecture)
- Results can require new special functions.

# Master integrals (cont.)

- For all-massless kinematics (e.g. for $gg \to gg$, $q\bar{q} \to q\bar{q}$, etc.), all you need is polylogarithms, $\mathrm{Li}_n(x)$, $n = 2, 3, 4$,

$$\mathrm{Li}_n(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^n} = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t), \qquad \mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

  where $-x \in \{s/t, t/s, s/u, u/s, t/u, u/t\}$.

- In any given channel, polylog identities relate all but 1 of $\mathrm{Li}_2(x)$, all but 2 of $\mathrm{Li}_3(x)$, all but 3 of $\mathrm{Li}_4(x)$.

- For more complicated kinematics — just one external mass — introduce 2-dimensional harmonic polylogs (2dHPLs) of weight 4. Unknown to `MAPLE, MATHEMATICA` $\Rightarrow$ must derive all analytical & numerical properties. Gehrmann & Remiddi

# Poles in $\epsilon$

- Programs for constructing the integrands, reducing them to master integrals, and Laurent expanding the master integrals, can be quite complicated (not quite pseudo-random!).
- It is therefore very useful to have cross checks. A crucial check is provided by the intricate structure of the infrared poles, which
  - begin at $1/\epsilon^4$
  - were organized up to $1/\epsilon$ by Catani
  - understood even further in terms of soft gluon resummation by Sterman & Tejeda-Yeomans.
- Phrased in terms of color-space notation: $|\mathcal{M}_n^{(L)}\rangle$ is treated as a vector over the possible color structures.

# Poles in $\epsilon$ (cont.)

- For example, for $gg \to gg$, there are 9 possible color structures (basis vectors), $|\mathcal{M}_n^{(L)}\rangle = \sum_{c=1}^{9} M_c^{(L)} \mathrm{Tr}^{[c]}$ with

$$\mathrm{Tr}^{[1]} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$$
$$\mathrm{Tr}^{[2]} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})$$
$$\mathrm{Tr}^{[3]} = \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})$$
$$\mathrm{Tr}^{[4]} = \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})$$
$$\mathrm{Tr}^{[5]} = \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})$$
$$\mathrm{Tr}^{[6]} = \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})$$
$$\mathrm{Tr}^{[7]} = \mathrm{Tr}(T^{a_1}T^{a_2})\mathrm{Tr}(T^{a_3}T^{a_4})$$
$$\mathrm{Tr}^{[8]} = \mathrm{Tr}(T^{a_1}T^{a_3})\mathrm{Tr}(T^{a_2}T^{a_4})$$
$$\mathrm{Tr}^{[9]} = \mathrm{Tr}(T^{a_1}T^{a_4})\mathrm{Tr}(T^{a_2}T^{a_3})$$

# Poles in $\epsilon$ (cont.)

- IR singularities of UV renormalized one-loop amplitudes:

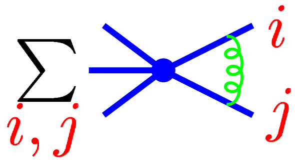
$$|\mathcal{M}_n^{(1)}\rangle = \boldsymbol{I}^{(1)}(\epsilon)\, |\mathcal{M}_n^{(0)}\rangle + |\mathcal{M}_n^{(1)\mathrm{fin}}\rangle$$

where

$$\boldsymbol{I}^{(1)}(\epsilon) = \frac{1}{2}\frac{e^{-\epsilon\psi(1)}}{\Gamma(1-\epsilon)} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \left[\frac{1}{\epsilon^2} + \frac{\gamma_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right]\left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}$$

- $1/\epsilon^2$ term is from soft/collinear overlap, only depends on color charge $\boldsymbol{T}_i$ of leg $i$. Pure collinear $1/\epsilon$ term depends on spin — $\gamma_q = 3/2\, C_F$, $\qquad \gamma_g = (11 C_A - 4 T_R N_f)/6$.

- Diagrammatically:



# Poles in $\epsilon$ (cont.)

- 2-loop IR poles:

$$|\mathcal{M}_n^{(2)}\rangle = \boldsymbol{I}^{(1)}(\epsilon)\,|\mathcal{M}_n^{(1)}\rangle + \boldsymbol{I}^{(2)}(\epsilon)\,|\mathcal{M}_n^{(0)}\rangle + |\mathcal{M}_n^{(2)\mathrm{fin}}\rangle$$

$$\begin{aligned}\boldsymbol{I}^{(2)}(\epsilon) &= -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\left(\boldsymbol{I}^{(1)}(\epsilon)+\frac{2b_0}{\epsilon}\right)+\frac{e^{\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{b_0}{\epsilon}+K\right)\boldsymbol{I}^{(1)}(2\epsilon) \\ &\quad \frac{e^{-\epsilon\psi(1)}}{4\epsilon\,\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-s}\right)^{2\epsilon}\left((n_q H_q^{(2)}+n_g H_g^{(2)})\,\mathbf{1}+\hat{\boldsymbol{H}}^{(2)}\right) \\ K &= \left[\frac{67}{18}-\frac{\pi^2}{6}-\left(\frac{1}{6}+\frac{4}{9}\epsilon\right)(1-\delta_R)\right]C_A-\frac{10}{9}T_R N_f\end{aligned}$$

$H_i^{(2)}$ and $\hat{\boldsymbol{H}}^{(2)}$ more complicated ...

- $\boldsymbol{I}^{(2)}(\epsilon)$ diagrammatically: